MA4110/MA5020: Computational Methods for Fluid Flow

Tutorial Sheet 3

Conservative Schemes, Finite Volume Methods, MUSCL, TVD and ENO (Lectures 9 to 13)

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Note. Read the lecture notes for Lectures 9 to 13 alongside this sheet. Several questions build directly on the derivations and examples in the notes, so work through the relevant section before attempting each question.

Notation. $\Delta_{i+1/2} = \bar{u}_{i+1} - \bar{u}_i$, $\nu = a\Delta t/\Delta x$, and Burgers' equation is $u_t + \left(u^2/2\right)_x = 0$.

Question 1: Conservative versus Non-Conservative Schemes

Consider Burgers' equation with the Riemann data

$$u(x,0) = \begin{cases} 2, & x \le 0, \\ 1, & x > 0, \end{cases}$$

on a uniform grid $x_j = j\Delta x$ with Neumann (zero gradient) boundary conditions. Take $\lambda = \Delta t/\Delta x = 0.25$.

(a) Find the exact shock speed.

(b) Advance the non-conservative upwind scheme $u_j^{n+1} = u_j^n - \lambda\, u_j^n \left(u_j^n - u_{j-1}^n\right)$ by two time steps. Tabulate u_j for $j = -1, \ldots, 3$.

(c) Repeat (b) with the conservative upwind scheme $u_j^{n+1} = u_j^n - \lambda\left(F_{j+1/2} - F_{j-1/2}\right)$, $F_{j+1/2} = \frac{1}{2}(u_j^n)^2$.

(d) Using the telescoping property, show that for the conservative scheme $\Delta x \sum_j u_j^n$ increases by exactly $\Delta t\big(f(2) - f(1)\big)$ per step, and explain why this forces the shock to travel at the Rankine-Hugoniot speed. Compute the increase per step for the non-conservative scheme from your table in (b) and compare.

Question 2: Numerical Fluxes and the Lax-Wendroff Theorem

(a) Show that the Lax-Friedrichs scheme $u_j^{n+1} = \frac{1}{2}\left(u_{j+1}^n + u_{j-1}^n\right) - \frac{\lambda}{2}\left(f(u_{j+1}^n) - f(u_{j-1}^n)\right)$ is conservative by finding its numerical flux $F_{j+1/2}$. Check that it is consistent.

(b) State the Lax-Wendroff theorem. Explain the two things it does *not* guarantee.

(c) Burgers' equation can also be written as $(u^2)_t + \left(\frac{2}{3}u^3\right)_x = 0$ for smooth solutions. Show that the two conservation forms give different shock speeds for the data of Question 1. Which one is physically correct, and why does this matter for the choice of conserved variable?

Question 3: Riemann Problems for Burgers' Equation

For each pair (u_L, u_R) below, (i) state whether the solution is a shock or a rarefaction, (ii) give the shock speed or the extent of the fan, (iii) find the state u^* at $x/t = 0$, and (iv) compute the Godunov flux $F^G = f(u^*)$ and the Rusanov flux

$$F^{\text{Rus}} = \tfrac{1}{2}\big(f(u_L) + f(u_R)\big) - \tfrac{1}{2}s_{\max}(u_R - u_L), \qquad s_{\max} = \max(|u_L|, |u_R|).$$

(a) $(3, 1)$ (b) $(1, 3)$ (c) $(-2, 1)$ (d) $(1, -3)$ (e) $(-1, -3)$

(a) Present the results in a table.

(b) Compute $F^{\mathrm{Rus}} - F^{G}$ in each case and show that it always has the form $-k(u_R - u_L)$ with $k > 0$. What does this say about the numerical dissipation of the Rusanov scheme?

Question 4: Hand Calculation: Godunov's Scheme for Burgers' Equation

Six cells of width $\Delta x = 0.1$ carry the averages

$$\bar{u}_1 = 1, \quad \bar{u}_2 = 1, \quad \bar{u}_3 = -1, \quad \bar{u}_4 = -1, \quad \bar{u}_5 = 2, \quad \bar{u}_6 = 2,$$

with transmissive (zero gradient) ghost cells. Take $\Delta t = 0.025$.

(a) Compute the Godunov flux at every interface. Identify the stationary shock and the transonic rarefaction.

(b) Advance one time step and tabulate $\bar{u}_i^1$. Check that $\sum_i \bar{u}_i \Delta x$ changes only by the boundary fluxes.

Question 5: Monotonicity of the Rusanov Scheme

Consider $u_t + f(u)_x = 0$ with the Rusanov flux and a constant dissipation coefficient $s_{\max} = \alpha > 0$, where $\alpha \geq \max |f'(u)|$.

(a) Write the update as $\bar{u}_i^{n+1} = \mathcal{H}(\bar{u}_{i-1}^n, \bar{u}_i^n, \bar{u}_{i+1}^n)$.

(b) Compute the three partial derivatives of $\mathcal{H}$ and show that the scheme is monotone if $\alpha \Delta t / \Delta x \leq 1$.

(c) State the Harten-Hyman-Lax result about the order of accuracy of monotone schemes. Why does it force us to look for nonlinear high-resolution schemes?

Question 6: MUSCL Reconstruction and Slope Limiters

Six cells carry the averages

$$\bar{u}_1 = 1, \quad \bar{u}_2 = 2, \quad \bar{u}_3 = 4, \quad \bar{u}_4 = 5, \quad \bar{u}_5 = 3, \quad \bar{u}_6 = 3.$$

(a) Compute $\Delta x\, \sigma_i$ for $i = 2, \ldots, 5$ using the centred slope $\frac{1}{2}(\Delta_{i-1/2} + \Delta_{i+1/2})$, the minmod slope, the MC slope

$$\Delta x\, \sigma_i = \operatorname{minmod}\left(2\Delta_{i-1/2},\ \tfrac{1}{2}\left(\Delta_{i-1/2} + \Delta_{i+1/2}\right),\ 2\Delta_{i+1/2}\right),$$

and the superbee slope

$$\Delta x\, \sigma_i = \operatorname{maxmod}\Big(\operatorname{minmod}\left(2\Delta_{i-1/2}, \Delta_{i+1/2}\right),\ \operatorname{minmod}\left(\Delta_{i-1/2}, 2\Delta_{i+1/2}\right)\Big).$$

(b) For each slope, compute the interface states $u^L_{i+1/2} = \bar{u}_i + \frac{1}{2}\Delta x\, \sigma_i$ and $u^R_{i-1/2} = \bar{u}_i - \frac{1}{2}\Delta x\, \sigma_i$. Which slopes create new extrema? Where?

(c) Show that the cell average of the reconstruction $\bar{u}_i + \sigma_i(x - x_i)$ over I_i is $\bar{u}_i$ for every choice of slope.

(d) How many ghost cells does MUSCL need at each boundary? Explain.

Question 7: Reconstruct-Evolve-Average for Linear Advection

Consider $u_t + a u_x = 0$ with $a > 0$ and $0 < \nu \leq 1$. In cell I_i the reconstruction is $\tilde{u}(x) = \bar{u}_i + \sigma_i(x - x_i)$.

(a) Use the exact translation of $\tilde{u}$ to find $u(x_{i+1/2}, t^n + \tau)$ for $0 \leq \tau \leq \Delta t$. Integrate in time to show that the exact time-averaged flux is

$$F_{i+1/2} = a\, \bar{u}_i + \frac{a}{2}(1 - \nu)\, \Delta x\, \sigma_i,$$

and write down the resulting update for $\bar{u}_i^{n+1}$.

(b) Show that the upwind slope $\Delta x\,\sigma_i = \Delta_{i-1/2}$ gives the Beam-Warming scheme

$$\bar{u}_i^{n+1} = \bar{u}_i - \frac{\nu}{2}\left(3\bar{u}_i - 4\bar{u}_{i-1} + \bar{u}_{i-2}\right) + \frac{\nu^2}{2}\left(\bar{u}_i - 2\bar{u}_{i-1} + \bar{u}_{i-2}\right).$$

(c) Take $\nu = 0.5$ and the cell averages

$$\bar{u}_1 = 0, \quad \bar{u}_2 = 0, \quad \bar{u}_3 = 1, \quad \bar{u}_4 = 2, \quad \bar{u}_5 = 2, \quad \bar{u}_6 = 0, \quad \bar{u}_7 = 0,$$

with zero gradient ghost cells. Advance one step with the zero slope, the upwind slope (Beam-Warming), the downwind slope $\Delta_{i+1/2}$ (Lax-Wendroff) and the minmod slope. Tabulate the four solutions.

(d) Which solutions leave the range $[0, 2]$ of the initial data, and in which cells? Which one is smeared the most? Relate your answers to the slopes each scheme uses next to the jumps.

Question 8: Harten's Lemma

A scheme in incremental form $\bar{u}_i^{n+1} = \bar{u}_i^n - C_{i-1/2}\Delta_{i-1/2} + D_{i+1/2}\Delta_{i+1/2}$ is TVD if $C_{i+1/2} \geq 0$, $D_{i+1/2} \geq 0$ and $C_{i+1/2} + D_{i+1/2} \leq 1$ for all i.

(a) Write the Lax-Friedrichs scheme for $u_t + au_x = 0$ in incremental form. For which ν is it TVD?

(b) Write the Lax-Wendroff scheme

$$\bar{u}_i^{n+1} = \bar{u}_i - \frac{\nu}{2}\left(\Delta_{i+1/2} + \Delta_{i-1/2}\right) + \frac{\nu^2}{2}\left(\Delta_{i+1/2} - \Delta_{i-1/2}\right)$$

in incremental form and show that Harten's conditions fail for $0 < \nu < 1$.

(c) Apply one step of Lax-Wendroff with $\nu = 0.5$ to the step data $\bar{u}_i = 1$ $(i \leq 0)$, $\bar{u}_i = 0$ $(i \geq 1)$. Show that the total variation increases.

Question 9: Flux Limiters and Sweby's TVD Region

For $u_t + au_x = 0$ with $a > 0$, $0 < \nu < 1$, consider

$$F_{i+1/2} = a\,\bar{u}_i + \frac{a}{2}(1-\nu)\,\phi(r_i)\,\Delta_{i+1/2}, \qquad r_i = \frac{\Delta_{i-1/2}}{\Delta_{i+1/2}}.$$

(a) Which $\phi(r)$ gives upwind, Lax-Wendroff and Beam-Warming?

(b) Write the scheme in incremental form and find $C_{i-1/2}$. Show that the scheme is TVD if $0 \leq \phi(r) \leq 2$ and $0 \leq \phi(r)/r \leq 2$, and deduce the Sweby region $\phi(r) = 0$ for $r \leq 0$, $0 \leq \phi(r) \leq \min(2r, 2)$ for $r > 0$.

(c) Show that the minmod slope corresponds to $\phi(r) = \max(0, \min(1, r))$.

(d) The second-order TVD region is $\min(r, 1) \leq \phi(r) \leq \max\big(\min(2r, 1), \min(r, 2)\big)$ for $r > 0$. Check that the MC limiter $\phi(r) = \max\big(0, \min\big(2r, \tfrac{1}{2}(1+r), 2\big)\big)$ and the van Leer limiter $\phi(r) = (r + |r|)/(1 + |r|)$ lie in it.

(e) Verify the symmetry property $\phi(r)/r = \phi(1/r)$ for minmod and van Leer.

(f) Explain why every TVD scheme clips smooth extrema.

Question 10: ENO Reconstruction

(a) Let $p(\xi) = a + b\xi + c\xi^2$ with $\xi = (x - x_i)/\Delta x$. Show that the average of p over cell I_{i+m} is $a + bm + c\left(m^2 + \frac{1}{12}\right)$. Using the stencil $\{i-2, i-1, i\}$, show that

$$p\left(\tfrac{1}{2}\right) = \tfrac{1}{3}\bar{u}_{i-2} - \tfrac{7}{6}\bar{u}_{i-1} + \tfrac{11}{6}\bar{u}_i.$$

(b) Consider the cell averages

$$\bar{u}_{i-2} = 0, \quad \bar{u}_{i-1} = 0, \quad \bar{u}_i = 2, \quad \bar{u}_{i+1} = 1, \quad \bar{u}_{i+2} = 1, \quad \bar{u}_{i+3} = 1.$$

Using third-order ENO, find the stencil chosen for $u^L_{i+1/2}$ and for $u^R_{i+1/2}$, and compute both values.

(c) For the same data, compute $u^L_{i+1/2}$ from the centred stencil $\{i-1, i, i+1\}$. Compare with (b).

(d) Second-order ENO uses the one-sided difference of smaller magnitude as the slope. Compare it with the minmod slope for the data $\bar{u}_{i-1} = 0$, $\bar{u}_i = 1$, $\bar{u}_{i+1} = 0.5$. Why is ENO not TVD, and why is this an advantage at smooth extrema?