MA4110/MA5020: Computational Methods for Fluid Flow

Tutorial Sheet 2

Finite Difference Approximations, Stability Analysis, and Numerical Errors

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Question 1: Hand Calculation: 1D Advection with a Step Profile

Consider the advection equation $\frac{\partial u}{\partial t} + 2\,\frac{\partial u}{\partial x} = 0$ on $x \in [0, 1]$ with wave speed $a = 2$. The initial condition is the Riemann data

$$u(x, 0) = \begin{cases} 1 & x \leq 0.4, \\ 0 & x > 0.4, \end{cases}$$

with inflow boundary condition $u(0, t) = 1$. Use $\Delta x = 0.2$ and $\Delta t = 0.05$.

(a) Set up the computational grid and write down the initial nodal values. Compute the Courant number and verify the stability condition for the first-order upwind scheme.

(b) Advance the solution two time steps using the first-order upwind scheme. Present your results in a table.

(c) Compare the numerical solution against the exact solution at $t = 0.10$. What feature of the exact solution is poorly represented, and why?

Question 2: Stability Analysis: Backward-Time Backward-Space (BTBS)

Consider the implicit upwind scheme for the advection equation $u_t + au_x = 0$ $(a > 0)$:

$$\frac{u_j^{n+1} - u_j^n}{\Delta t} + a\frac{u_j^{n+1} - u_{j-1}^{n+1}}{\Delta x} = 0$$

(a) Substitute a Fourier mode $u_j^n = \hat{u}^n e^{\mathrm{i}j\xi}$ and derive the amplification factor $G(\xi)$ in terms of $\nu = \frac{a\Delta t}{\Delta x}$ and $\xi = k\Delta x$.

(b) Show that $|G(\xi)|^2 = \dfrac{1}{(1 + \nu - \nu\cos\xi)^2 + (\nu\sin\xi)^2}$.

(c) Prove that $|G(\xi)| \leq 1$ for all $\xi \in [-\pi, \pi]$ for **any** $\nu > 0$. What does this imply about the stability restriction on Δt?

Question 3: Stability Analysis: The Leapfrog (CTCS) Scheme

The Leapfrog scheme for $u_t + au_x = 0$ is central in both time and space:

$$\frac{u_j^{n+1} - u_j^{n-1}}{2\Delta t} + a\frac{u_{j+1}^n - u_{j-1}^n}{2\Delta x} = 0$$

(a) Substitute $u_j^n = G^n e^{\mathrm{i}j\xi}$ and show that G satisfies: $G^2 + 2\mathrm{i}\nu\sin\xi\, G - 1 = 0$.

(b) Solve for the roots $G_{1,2} = -\mathrm{i}\nu\sin\xi \pm \sqrt{1 - \nu^2\sin^2\xi}$.

(c) Show that when $|\nu| \leq 1$, both roots satisfy $|G| = 1$ identically. What does $|G| = 1$ imply about numerical dissipation?

Question 4: Modified Equation: First-Order Upwind Scheme

The first-order upwind scheme for $u_t + au_x = 0$ $(a > 0)$ is

$$\frac{u_j^{n+1} - u_j^n}{\Delta t} + a\frac{u_j^n - u_{j-1}^n}{\Delta x} = 0.$$

(a) Expand u_j^{n+1} and u_{j-1}^n as Taylor series about (x_j, t^n) to second order in Δt and Δx. Show that the scheme can be written as

$$u_t + au_x = \frac{a\Delta x}{2}(1-\nu)\, u_{xx} + O(\Delta x^2, \Delta t^2),$$

where $\nu = a\Delta t/\Delta x$ is the Courant number.

(b) Identify the leading error term. What kind of physical process does it mimic, and what is the effective numerical diffusion coefficient κ_{num}?

(c) Show that the modified equation implies the scheme is first-order accurate in both space and time. Under what condition on ν does numerical diffusion vanish, and is this condition practically achievable?

Question 5: Stability Analysis: Lax–Wendroff Scheme

The Lax–Wendroff scheme for the advection equation $u_t + au_x = 0$ is:

$$u_j^{n+1} = u_j^n - \frac{\nu}{2}\left(u_{j+1}^n - u_{j-1}^n\right) + \frac{\nu^2}{2}\left(u_{j+1}^n - 2u_j^n + u_{j-1}^n\right),$$

where $\nu = a\Delta t/\Delta x$.

(a) Substitute the Fourier mode $u_j^n = \hat{u}^n e^{\mathrm{i}j\xi}$ and show that the amplification factor is

$$G(\xi) = 1 - \nu^2(1-\cos\xi) - \mathrm{i}\,\nu\sin\xi.$$

(b) Compute $|G(\xi)|^2$ and show that

$$|G(\xi)|^2 = 1 - \nu^2(1-\nu^2)(1-\cos\xi)^2.$$

(c) Hence prove the scheme is stable if and only if $|\nu| \le 1$.
Hint: Show $|G|^2 \le 1$ requires $\nu^2(1-\nu^2) \ge 0$.

(d) Perform a modified equation analysis for Lax–Wendroff. Show that the leading truncation error term is dispersive rather than diffusive:

$$u_t + au_x = -\frac{a(\Delta x)^2}{6}(1-\nu^2)\, u_{xxx} + O(\Delta x^3).$$

What is the formal order of accuracy?

Question 6: Hand Calculation: 1D Advection with Periodic Boundary Conditions

Consider the advection equation $u_t + u_x = 0$ on the periodic domain $x \in [0,1)$ with initial condition $u(x,0) = \sin(2\pi x)$. Use $N = 4$ uniformly spaced grid points and choose Δt so that the Courant number is $\nu = 1$.

(a) Set up the grid, tabulate the initial nodal values, and verify that the chosen Δt satisfies the stability condition for the first-order upwind scheme.

(b) Advance the solution one time step using the first-order upwind scheme. Compare with the exact solution and comment on the result.

(c) Repeat part (b) using the Lax–Wendroff scheme. Compare the two numerical solutions and explain any differences.

Question 7: Modified Equations: Dissipation vs. Dispersion

Consider the advection equation $u_t + au_x = 0$ with $a > 0$.

(a) **First-order upwind.** Perform a modified equation analysis by expanding u_j^{n+1} and u_{j-1}^n as Taylor series about (x_j, t^n). Show that the scheme satisfies

$$u_t + au_x = \underbrace{\frac{a\Delta x}{2}(1-\nu)}_{\kappa_{\text{num}}} u_{xx} + O(\Delta x^2),$$

where $\nu = a\Delta t/\Delta x$. Identify κ_{num} as a numerical diffusion coefficient, and state for which values of $\nu \in (0,1]$ it is non-negative.

(b) **Lax–Wendroff.** The modified equation for Lax–Wendroff is

$$u_t + au_x = -\frac{a(\Delta x)^2}{6}(1-\nu^2)\, u_{xxx} + O(\Delta x^3).$$

Without re-deriving this, explain why the leading error is a third-derivative (dispersive) term rather than a second-derivative (diffusive) term. What physical artifact does dispersion error produce in the numerical solution?

(c) **Comparison.** Fill in the table below, where "sign of leading error" refers to the sign of the dominant term for $0 < \nu < 1$:

Scheme	**Order**	**Error type**	**Effect on solution**
First-order Upwind	$O(\Delta x)$	Diffusive	
Lax–Wendroff	$O(\Delta x^2)$	Dispersive	
Leapfrog (CTCS)	$O(\Delta x^2)$		

Complete the missing entries and briefly justify each.