MA4110/MA5020: Computational Methods for Fluid Flow

Tutorial Sheet 1

Classification of PDEs, Method of Characteristics, and Shock Formation

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1. For a general second-order PDE $Au_{xx} + Bu_{xy} + Cu_{yy} + Du_x + Eu_y + Fu = G$, recall that the type is decided entirely by the discriminant $\Delta = B^2 - 4AC$. Classify each of the following as hyperbolic ($\Delta > 0$), parabolic ($\Delta = 0$), or elliptic ($\Delta < 0$) in the regions of the (x, y) plane where they are defined. Where the type changes, identify the boundary curve separating the regimes:

   (a) $x\, u_{xx} + u_{yy} = 0$ (Tricomi-type equation)

   (b) $u_{xx} + 2x\, u_{xy} + (1 - y^2)\, u_{yy} = 0$

   (c) $u_{xx} + 4u_{xy} + 4u_{yy} + 2u_x - u = 0$

2. Recall the classification hierarchy for first-order PDEs of the general form $F(x, t, u, u_x, u_t) = 0$:

   - **Linear:** Linear in u, u_x, u_t, with coefficients depending only on (x, t):

   $$a(x,t)\, u_t + b(x,t)\, u_x + c(x,t)\, u = d(x,t)$$

   - **Semilinear:** Linear in the highest derivatives (u_t, u_x) with coefficients depending only on (x, t), while non-linearities enter only through lower-order terms:

   $$a(x,t)\, u_t + b(x,t)\, u_x = f(x,t,u)$$

   - **Quasilinear:** Linear in the highest derivatives (u_t, u_x), but their coefficients may depend on u as well as (x, t):

   $$a(x,t,u)\, u_t + b(x,t,u)\, u_x = f(x,t,u)$$

   - **Fully Nonlinear:** Non-linear in the highest derivatives (u_t, u_x), e.g., $(u_x)^2$, $\sqrt{1 + u_x^2}$, etc.

   Classify each first-order PDE below into one of these four categories and justify your choice:

   (a) $u_t + c\, u_x = 0$, c a constant

   (b) $u_t + x\, u_x = t$

   (c) $u_t + u\, u_x = 0$ (Inviscid Burgers' equation)

   (d) $u_t + u_x = u^2$

   (e) $u_t + u^2\, u_x = 0$

   (f) $(u_x)^2 + u_t = 0$ (Hamilton-Jacobi type equation)

   (g) $u_t + \sin(x)\, u_x + u = 0$

   (h) $u_t + \cos(u)\, u_x = 0$ (Traffic flow model with density-dependent velocity)

   (i) $a(x,t)\, u_x + b(x,t)\, u_t + c(x,t)\, u = f(x,t)$

(j) $u_t + \sqrt{1 + (u_x)^2} = 0$ (Eikonal / front propagation equation)

3. (a) Solve the initial value problem

$$\frac{\partial u}{\partial t} + (1 + x^2)\frac{\partial u}{\partial x} = 0, \qquad x \in \mathbb{R},\ t > 0,$$

subject to $u(x, 0) = \dfrac{1}{1 + x^2}$.

i. Write down the characteristic ODE and find the characteristic curve $x(t; x_0)$ passing through $(x_0, 0)$.

ii. Show that a characteristic starting at finite x_0 reaches $x \to +\infty$ in finite time, and determine this transit time.

iii. Determine the explicit closed-form solution $u(x, t)$.

(b) Consider the constant-coefficient advection equation $u_t + 2u_x = 0$ with initial data $u(x, 0) = u_0(x)$.

i. Find the characteristic line through the point $(x, t) = (5, 2)$, and state the **domain of dependence** of this point.

ii. If the initial data is perturbed only on the interval $x_0 \in [0, 1]$, describe the **domain of influence** of this interval, and sketch the affected region of the (x, t) plane at time $t = 3$.

4. For each PDE and initial data below, find the characteristic curves $x(t; x_0)$, sketch a family of them for representative values of x_0 in the (x, t) half-plane $t \geq 0$, and answer the question posed:

(a) $u_t + u_x = 0$, $u(x, 0) = u_0(x)$. Sketch the characteristics for $x_0 = -2, -1, 0, 1, 2$. Since these are parallel straight lines, explain why the solution is a rigid translation of u_0, and mark on your sketch the point reached at $t = 3$ by the characteristic starting at $x_0 = -1$.

(b) $u_t + x\,u_x = 0$, $u(x, 0) = u_0(x)$. Show that the characteristic ODE is $\frac{dx}{dt} = x$, solve it, and sketch the characteristics for $x_0 = -2, -1, -0.5, 0.5, 1, 2$. What happens along the line $x_0 = 0$, and how does the spacing between neighboring characteristics evolve as t increases?

(c) $u_t + t\,u_x = 0$, $u(x, 0) = u_0(x)$. Find $x(t; x_0)$ (noting that the wave speed depends explicitly on t), and sketch the resulting parabolic characteristic curves for $x_0 = -2, -1, 0, 1, 2$.

(d) $u_t + u\,u_x = 0$ with $u_0(x) = x$ for $x \in [0, 1]$ and $u_0(x) = 0$ elsewhere. Sketch the characteristics for $x_0 = 0, 0.25, 0.5, 0.75, 1$ along with the outer constant regions. Do any characteristics cross for $t > 0$? What kind of wave structure does this expanding fan represent?

(e) $u_t + u\,u_x = 0$ with $u_0(x) = 1$ for $x < 0$ and $u_0(x) = 0$ for $x > 0$. Sketch the characteristics from both sides of $x_0 = 0$ and mark the region where they collide for $t > 0$. Explain why a discontinuity (shock) must be introduced to resolve the multi-valued state.

5. Consider the inviscid Burgers' equation $u_t + uu_x = 0$ with smooth initial data $u(x, 0) = u_0(x) = \dfrac{1}{1 + x^2}$.

(a) Write down the characteristic lines $x(t; x_0) = x_0 + u_0(x_0)\,t$.

(b) Compute $u_0'(x)$ and identify the regions of compression ($u_0' < 0$) and expansion ($u_0' > 0$).

(c) Using the derivative evolution formula $\frac{\partial u}{\partial x} = \dfrac{u_0'(x_0)}{1 + u_0'(x_0)t}$, find $\min_x u_0'(x)$ and determine the exact breaking time $t_s = \dfrac{-1}{\min_x u_0'(x)}$.

(d) Find the spatial location $x_s = x_0^* + u_0(x_0^*)\, t_s$ where the vertical tangent first appears.

6. For the inviscid Burgers' equation $u_t + uu_x = 0$ with initial condition $u(x,0) = u_0(x)$, a classical smooth solution breaks down and forms a shock at the breaking time:

$$t_s = \frac{-1}{\min\limits_x u_0'(x)}, \qquad \text{provided } \min_x u_0'(x) < 0.$$

For each initial profile below, find $u_0'(x)$, determine $\min_x u_0'(x)$ (and the point x_0^* where the minimum slope occurs), and compute the breaking time t_s and breaking coordinate x_s:

(a) $u_0(x) = -\sin(x)$

(b) $u_0(x) = -\tanh(x)$

(c) $u_0(x) = \begin{cases} 1, & x \le 0 \\ 1 - x, & 0 < x < 1 \\ 0, & x \ge 1 \end{cases}$ (Linear ramp profile)

(d) $u_0(x) = -x\, e^{-x^2}$

(e) $u_0(x) = \dfrac{2}{1 + x^2}$