MA4110/MA5020: Computational Methods for Fluid Flow

Lecture 13: ENO and WENO Schemes

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1 Introduction: Beyond TVD

Lecture 12 ended with a negative result. Every TVD limiter sets $\phi(r) = 0$ at an extremum, so a TVD scheme is first order at every smooth maximum and minimum (Osher and Chakravarthy), and in two dimensions a TVD scheme is first order everywhere (Goodman and LeVeque). We saw the effect in the convergence table: the L^∞ order of the limited schemes dropped to about 1.3.

The root of the problem is that TVD treats a smooth extremum and a discontinuity alike: both make $r \le 0$ or r far from 1, and both are cured by throwing away the slope. To do better we need a reconstruction that can *tell the two apart*. This lecture develops two such reconstructions:

1. **ENO** (Essentially Non-Oscillatory; Harten, Engquist, Osher and Chakravarthy, 1987): build a polynomial of higher degree, and choose its stencil adaptively so that it avoids discontinuities.
2. **WENO** (Weighted ENO; Liu, Osher and Chan, 1994; Jiang and Shu, 1996): instead of choosing one stencil, take a weighted average of all candidate stencils, with weights that switch off any stencil that crosses a discontinuity.

Both drop the TVD requirement. They only ask that the scheme be *essentially* non-oscillatory: spurious oscillations, if they occur at all, are of the size of the truncation error and vanish as $\Delta x \to 0$.

The framework is the semi-discrete finite volume scheme of Lecture 11,

$$\frac{d\bar{u}_i}{dt} = -\frac{1}{\Delta x}\Big[\mathcal{F}\big(u^L_{i+1/2}, u^R_{i+1/2}\big) - \mathcal{F}\big(u^L_{i-1/2}, u^R_{i-1/2}\big)\Big], \tag{1}$$

with a monotone numerical flux $\mathcal{F}$ (Godunov or Rusanov) and SSP-RK3 in time (Lecture 12). Only one ingredient changes: how the interface states $u^L_{i+1/2}$ and $u^R_{i+1/2}$ are computed from the cell averages. Godunov used piecewise constants, MUSCL used limited piecewise linears; ENO and WENO use piecewise *quadratics* (or higher).

2 Reconstruction from Cell Averages

2.1 The Problem

We are given cell averages $\bar{u}_j = \frac{1}{\Delta x}\int_{I_j} u\,dx$ on a uniform grid with cells $I_j = [x_{j-1/2}, x_{j+1/2}]$. We want a value of u at the interface $x_{i+1/2}$, accurate to high order. A point value cannot be read off from a single average, so we use several neighbouring cells: a **stencil**.

Reconstruction Problem

Given a stencil of k consecutive cells, find the polynomial $p(x)$ of degree $k-1$ whose cell averages agree with the data on every cell of the stencil:

$$\frac{1}{\Delta x}\int_{I_j} p(x)\,dx = \bar{u}_j \qquad \text{for every } j \text{ in the stencil.} \tag{2}$$

Then use $p(x_{i+1/2})$ as the interface value. If u is smooth, $p(x_{i+1/2}) = u(x_{i+1/2}) + \mathcal{O}(\Delta x^k)$.

For $k = 1$ this is Godunov's piecewise constant reconstruction, and for $k = 2$ it is a linear reconstruction as in MUSCL. Note that we match *averages*, not point values: this is what keeps the scheme conservative, as in Lecture 11.

2.2 Quadratic Reconstruction: A Worked Derivation

Take $k = 3$. Around cell i there are three stencils of three cells that contain I_i:

$$S_0 = \{i-2,\ i-1,\ i\}, \qquad S_1 = \{i-1,\ i,\ i+1\}, \qquad S_2 = \{i,\ i+1,\ i+2\}. \tag{3}$$

S_0 is biased to the left, S_1 is centred and S_2 is biased to the right. On each stencil we look for a quadratic. It is convenient to use the local coordinate $\xi = (x - x_i)/\Delta x$, so that cell I_{i+m} is the interval $[m - \frac{1}{2}, m + \frac{1}{2}]$ in ξ, and the interface $x_{i+1/2}$ is $\xi = \frac{1}{2}$. Write

$$p(\xi) = a + b\,\xi + c\,\xi^2. \tag{4}$$

The averages of the three monomials over cell I_{i+m} are

$$\int_{m-1/2}^{m+1/2} 1\,d\xi = 1, \qquad \int_{m-1/2}^{m+1/2} \xi\,d\xi = m, \qquad \int_{m-1/2}^{m+1/2} \xi^2\,d\xi = m^2 + \frac{1}{12},$$

so condition (2) on cell I_{i+m} reads

$$\bar{u}_{i+m} = a + b\,m + c\left(m^2 + \tfrac{1}{12}\right). \tag{5}$$

The centred stencil S_1 $(m = -1, 0, 1)$. From (5),

$$\bar{u}_{i-1} = a - b + \tfrac{13}{12}c, \qquad \bar{u}_i = a + \tfrac{1}{12}c, \qquad \bar{u}_{i+1} = a + b + \tfrac{13}{12}c.$$

Subtracting and adding the outer equations gives

$$b_1 = \frac{\bar{u}_{i+1} - \bar{u}_{i-1}}{2}, \qquad c_1 = \frac{\bar{u}_{i-1} - 2\bar{u}_i + \bar{u}_{i+1}}{2}, \qquad a_1 = \bar{u}_i - \frac{c_1}{12}. \tag{6}$$

The interface value is

$$p_1(\tfrac{1}{2}) = a_1 + \frac{b_1}{2} + \frac{c_1}{4} = \bar{u}_i + \frac{b_1}{2} + \frac{c_1}{6} = -\frac{1}{6}\,\bar{u}_{i-1} + \frac{5}{6}\,\bar{u}_i + \frac{1}{3}\,\bar{u}_{i+1}.$$

All three stencils. The same computation on S_0 $(m = -2, -1, 0)$ and S_2 $(m = 0, 1, 2)$ is Exercise E1. In every case $a_k = \bar{u}_i - c_k/12$, because cell i belongs to all three stencils, and the interface value is $p_k(\frac{1}{2}) = \bar{u}_i + \frac{1}{2}b_k + \frac{1}{6}c_k$. The results are collected below.

Third-Order Reconstructions of $u^L_{i+1/2}$

Stencil	**Slope b_k**	**Curvature c_k**	$u^{(k)}_{i+1/2} = p_k(x_{i+1/2})$
S_0	$\frac{1}{2}(\bar{u}_{i-2} - 4\bar{u}_{i-1} + 3\bar{u}_i)$	$\frac{1}{2}(\bar{u}_{i-2} - 2\bar{u}_{i-1} + \bar{u}_i)$	$\frac{1}{3}\bar{u}_{i-2} - \frac{7}{6}\bar{u}_{i-1} + \frac{11}{6}\bar{u}_i$
S_1	$\frac{1}{2}(\bar{u}_{i+1} - \bar{u}_{i-1})$	$\frac{1}{2}(\bar{u}_{i-1} - 2\bar{u}_i + \bar{u}_{i+1})$	$-\frac{1}{6}\bar{u}_{i-1} + \frac{5}{6}\bar{u}_i + \frac{1}{3}\bar{u}_{i+1}$
S_2	$\frac{1}{2}(-3\bar{u}_i + 4\bar{u}_{i+1} - \bar{u}_{i+2})$	$\frac{1}{2}(\bar{u}_i - 2\bar{u}_{i+1} + \bar{u}_{i+2})$	$\frac{1}{3}\bar{u}_i + \frac{5}{6}\bar{u}_{i+1} - \frac{1}{6}\bar{u}_{i+2}$

Each $u^{(k)}_{i+1/2}$ is third-order accurate when u is smooth on the stencil S_k.

The names are meaningful. A Taylor expansion of the cell averages (Exercise E2) shows that, for smooth u,

$$b_k = \Delta x\,u'(x_i) + \mathcal{O}(\Delta x^3), \qquad c_k = \tfrac{1}{2}\Delta x^2\,u''(x_i) + \mathcal{O}(\Delta x^3). \tag{7}$$

So b_k is the jump of u across one cell and c_k measures its curvature. Near a discontinuity both become $\mathcal{O}(1)$ if S_k contains it.

The right state $u^R_{i+1/2}$ is the value at $x_{i+1/2}$ of a reconstruction in cell $i+1$. By symmetry it is obtained from the same formulas applied to the mirrored data: replace $(\bar{u}_{i-2}, \bar{u}_{i-1}, \bar{u}_i, \bar{u}_{i+1}, \bar{u}_{i+2})$ by $(\bar{u}_{i+3}, \bar{u}_{i+2}, \bar{u}_{i+1}, \bar{u}_i, \bar{u}_{i-1})$.

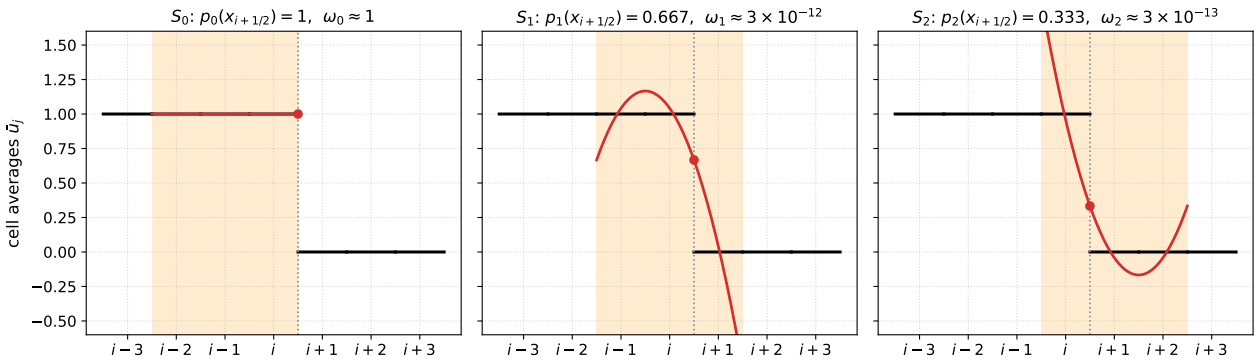


Figure 1: The three candidate quadratics for $u^L_{i+1/2}$ next to a step between cells i and $i+1$. Black: cell averages. Red: the parabola p_k with the right averages on the shaded stencil S_k, and its value at $x_{i+1/2}$ (dotted line). Only S_0 avoids the step. The WENO weights ω_k (Section 4) automatically select it.

2.3 Which Stencil?

In a smooth region all three stencils give third order. Near a discontinuity, any stencil that contains it gives an interpolating parabola that overshoots wildly. Figure 1 shows a step between cells i and $i+1$. Only S_0 lies entirely on one side of the step, and only S_0 gives the correct value $u^L_{i+1/2} = 1$. The centred stencil S_1, which is the natural choice for smooth data, gives $2/3$, and S_2 gives $1/3$. Choosing a single fixed stencil, as the classical schemes do, is what produces Gibbs-type oscillations (Lecture 11). ENO and WENO are two answers to the question *which stencil?*

3 ENO: Choose the Smoothest Stencil

3.1 The Idea

Harten, Engquist, Osher and Chakravarthy (1987) proposed to build the stencil one cell at a time, starting from cell i itself, and each time to extend it in the direction where the data are smoother. Smoothness is measured by the size of differences: across a discontinuity the differences are $\mathcal{O}(1)$, while in smooth regions the m-th difference is $\mathcal{O}(\Delta x^m)$.

ENO Stencil Selection (third order)

Step 1. Start with the one-cell stencil $\{i\}$.

Step 2. First differences. Compare $|\bar{u}_i - \bar{u}_{i-1}|$ and $|\bar{u}_{i+1} - \bar{u}_i|$. Add the cell on the side with the smaller difference:

$$\{i-1, i\} \text{ if } |\bar{u}_i - \bar{u}_{i-1}| < |\bar{u}_{i+1} - \bar{u}_i|, \qquad \{i, i+1\} \text{ otherwise.}$$

Step 3. Second differences. Let the current stencil be $\{j, j+1\}$. Compare the second differences of the two possible extensions,

$$|\bar{u}_{j-1} - 2\bar{u}_j + \bar{u}_{j+1}| \quad (\text{add } j-1) \qquad \text{and} \qquad |\bar{u}_j - 2\bar{u}_{j+1} + \bar{u}_{j+2}| \quad (\text{add } j+2),$$

and add the cell with the smaller one.

The result is one of S_0, S_1, S_2, and $u^L_{i+1/2}$ is the corresponding value from the table above.

Higher-order ENO continues in the same way with third, fourth, ... differences. Formally, the selection is done on the divided differences of the primitive function $U(x) = \int_{x_{1/2}}^{x} u\, d\xi$, whose values

at the interfaces are known exactly from the cell averages; on a uniform grid this is equivalent to comparing the undivided differences of $\bar{u}$ above.

Example. For the step of Figure 1, $\bar{u}_{i-2} = \bar{u}_{i-1} = \bar{u}_i = 1$ and $\bar{u}_{i+1} = \bar{u}_{i+2} = 0$. Step 2: $|\bar{u}_i - \bar{u}_{i-1}| = 0 < 1 = |\bar{u}_{i+1} - \bar{u}_i|$, so the stencil becomes $\{i-1, i\}$. Step 3 with $j = i-1$: $|\bar{u}_{i-2} - 2\bar{u}_{i-1} + \bar{u}_i| = 0 < 1 = |\bar{u}_{i-1} - 2\bar{u}_i + \bar{u}_{i+1}|$, so the stencil becomes $\{i-2, i-1, i\} = S_0$. ENO finds the stencil that avoids the step, and $u^L_{i+1/2} = 1$.

3.2 ENO at Smooth Extrema

Consider second-order ENO, that is, stop after Step 2. Its slope is the one-sided difference of smaller magnitude. Compare this with minmod from Lecture 11:

$$\Delta x\, \sigma_i^{\text{ENO}} = \begin{cases} \Delta_{i-1/2} & \text{if } |\Delta_{i-1/2}| < |\Delta_{i+1/2}|, \\ \Delta_{i+1/2} & \text{otherwise,} \end{cases} \qquad \Delta x\, \sigma_i^{\text{minmod}} = \begin{cases} \text{the same} & \text{if } \Delta_{i-1/2}\Delta_{i+1/2} > 0, \\ 0 & \text{otherwise.} \end{cases}$$

The only difference is at an extremum, where the two jumps have opposite signs. Minmod sets the slope to zero (clipping); ENO keeps the smaller of the two one-sided slopes, which is still a consistent approximation of u'. This single change is why ENO does not clip extrema. The price is that ENO is *not* TVD: near an extremum it can create a small new overshoot. The theory (and practice) of ENO is that such overshoots are of the size of the truncation error, $\mathcal{O}(\Delta x^k)$, which is what "essentially non-oscillatory" means.

3.3 Weaknesses of ENO

ENO works, and third-order ENO is visibly sharper than MUSCL (Section 6). But it has three drawbacks.

(1) Wasted information. The selection examines $2k-1$ cells (here $i-2, \ldots, i+2$) but uses only k of them. A smarter combination of the same five cells could give fifth order instead of third.

(2) Unstable selection in smooth regions. When two differences are almost equal, as happens where $u'' \approx 0$, a round-off-sized perturbation can flip the choice. The stencil then jumps back and forth between time steps, which can reduce accuracy.

(3) A non-smooth flux. The "$<$" in the selection makes the numerical flux a discontinuous function of the data. This complicates analysis and steady-state convergence.

WENO removes all three.

4 WENO: Weight the Stencils

4.1 The Optimal Linear Combination

The five cells $i-2, \ldots, i+2$ support a single degree-4 polynomial, and its value at $x_{i+1/2}$ is fifth-order accurate. Is this value a combination of the three third-order values? Look for constants d_0, d_1, d_2 with

$$d_0\, u^{(0)}_{i+1/2} + d_1\, u^{(1)}_{i+1/2} + d_2\, u^{(2)}_{i+1/2} = \frac{1}{60}\left(2\bar{u}_{i-2} - 13\bar{u}_{i-1} + 47\bar{u}_i + 27\bar{u}_{i+1} - 3\bar{u}_{i+2}\right), \tag{8}$$

where the right-hand side is the fifth-order value (Exercise E1 derives it). Only $u^{(0)}$ contains $\bar{u}_{i-2}$, so matching that coefficient gives $\frac{1}{3}d_0 = \frac{2}{60}$. Only $u^{(2)}$ contains $\bar{u}_{i+2}$, so $-\frac{1}{6}d_2 = -\frac{3}{60}$. Consistency requires the weights to sum to one. Hence

$$\boxed{d_0 = \frac{1}{10}, \qquad d_1 = \frac{6}{10}, \qquad d_2 = \frac{3}{10}.} \tag{9}$$

One checks that the remaining coefficients of $\bar{u}_{i-1}$, $\bar{u}_i$, $\bar{u}_{i+1}$ then match automatically. The d_k are called the **linear** or **optimal weights**. They are all positive, so the combination is a convex combination.

Using the d_k as they are gives a fixed, linear, fifth-order upwind-biased scheme. By Godunov's Barrier Theorem (Lecture 8) it must oscillate at discontinuities. The idea of WENO is to keep the form of a convex combination but to make the weights *depend on the data*:

$$\boxed{u^L_{i+1/2} = \omega_0\, u^{(0)}_{i+1/2} + \omega_1\, u^{(1)}_{i+1/2} + \omega_2\, u^{(2)}_{i+1/2}, \qquad \omega_k \geq 0, \quad \omega_0 + \omega_1 + \omega_2 = 1,} \tag{10}$$

with two requirements:

- in smooth regions, $\omega_k \approx d_k$, so that we get fifth order;
- if S_k contains a discontinuity, $\omega_k \approx 0$, so that the result behaves like ENO.

4.2 Smoothness Indicators

To build the weights we need a number β_k that is small when p_k is smooth and large when it is not. Jiang and Shu (1996) proposed the scaled total squared variation of p_k over cell I_i:

$$\beta_k = \sum_{l=1}^{2} \Delta x^{2l-1} \int_{I_i} \left(\frac{d^l p_k}{dx^l} \right)^2 dx. \tag{11}$$

The powers of Δx make β_k independent of the grid scale. Let us compute it in the local coordinate ξ, where $dx = \Delta x\, d\xi$ and $d/dx = \Delta x^{-1}\, d/d\xi$. With $p_k = a_k + b_k\xi + c_k\xi^2$:

$$\Delta x \int_{I_i} \left(p_k'\right)^2 dx = \int_{-1/2}^{1/2} (b_k + 2c_k\xi)^2\, d\xi = b_k^2 + \frac{c_k^2}{3}, \qquad \Delta x^3 \int_{I_i} \left(p_k''\right)^2 dx = \int_{-1/2}^{1/2} (2c_k)^2\, d\xi = 4c_k^2.$$

(The cross term $4b_kc_k\xi$ integrates to zero.) Adding,

$$\boxed{\beta_k = b_k^2 + \frac{13}{3}\, c_k^2} \tag{12}$$

With b_k and c_k from the table, this is the familiar form

$$\begin{aligned}
\beta_0 &= \tfrac{13}{12} \left(\bar u_{i-2} - 2\bar u_{i-1} + \bar u_i\right)^2 + \tfrac{1}{4} \left(\bar u_{i-2} - 4\bar u_{i-1} + 3\bar u_i\right)^2, \\
\beta_1 &= \tfrac{13}{12} \left(\bar u_{i-1} - 2\bar u_i + \bar u_{i+1}\right)^2 + \tfrac{1}{4} \left(\bar u_{i-1} - \bar u_{i+1}\right)^2, \\
\beta_2 &= \tfrac{13}{12} \left(\bar u_i - 2\bar u_{i+1} + \bar u_{i+2}\right)^2 + \tfrac{1}{4} \left(3\bar u_i - 4\bar u_{i+1} + \bar u_{i+2}\right)^2.
\end{aligned} \tag{13}$$

By (7), in a smooth region

$$\beta_k = \Delta x^2\, u'(x_i)^2 + \mathcal{O}(\Delta x^4), \tag{14}$$

the same leading term for all three stencils. If S_k contains a discontinuity, $\beta_k = \mathcal{O}(1)$.

4.3 Nonlinear Weights

The WENO weights are

$$\boxed{\alpha_k = \frac{d_k}{(\varepsilon + \beta_k)^2}, \qquad \omega_k = \frac{\alpha_k}{\alpha_0 + \alpha_1 + \alpha_2}} \tag{15}$$

where ε is a small number (typically 10^{-6}) that prevents division by zero. By construction $\omega_k \geq 0$ and $\sum \omega_k = 1$. Let us check the two requirements.

Smooth region. If $u'(x_i) \neq 0$, write $\beta_k = D\,(1 + \mathcal{O}(\Delta x^2))$ with $D = \Delta x^2 u'(x_i)^2$, from (14). Ignoring ε, $\alpha_k = \frac{d_k}{D^2}(1 + \mathcal{O}(\Delta x^2))$. The common factor $1/D^2$ cancels in the normalisation, and

$$\omega_k = d_k + \mathcal{O}(\Delta x^2).$$

The weights are close to the optimal ones, and one can show that this is enough for fifth-order accuracy.

Near a discontinuity. Suppose S_0 is smooth but S_1 and S_2 contain a jump. Then $\beta_0 = \mathcal{O}(\Delta x^2)$ and $\beta_1, \beta_2 = \mathcal{O}(1)$, so

$$\frac{\alpha_1}{\alpha_0} = \frac{d_1}{d_0}\left(\frac{\varepsilon + \beta_0}{\varepsilon + \beta_1}\right)^2 = \mathcal{O}(\Delta x^4),$$

and similarly for α_2. Hence $\omega_0 \approx 1$ and $\omega_1, \omega_2 = \mathcal{O}(\Delta x^4)$: WENO reduces to ENO and uses the smooth stencil. The square in (15) is what makes this switch so sharp.

Example. For the step data $(1, 1, 1, 0, 0)$, equation (13) gives

$$\beta_0 = 0, \qquad \beta_1 = \tfrac{13}{12} + \tfrac{1}{4} = \tfrac{4}{3}, \qquad \beta_2 = \tfrac{13}{12} + \tfrac{9}{4} = \tfrac{10}{3}.$$

With $\varepsilon = 10^{-6}$: $\alpha_0 = 0.1/10^{-12} = 10^{11}$, $\alpha_1 = 0.6/(4/3)^2 \approx 0.34$, $\alpha_2 = 0.3/(10/3)^2 = 0.027$. So $\omega_0 = 1 - 3.6 \times 10^{-12}$, and $u^L_{i+1/2} = 1$ to twelve digits (Figure 1).

Insight: ENO Chooses, WENO Weighs

ENO makes a hard, discontinuous decision: one stencil wins. WENO makes a soft, smooth decision: every stencil contributes, in proportion to its smoothness. In smooth regions the soft decision combines all five cells into a fifth-order value, so with the same $2k-1$ cells WENO reaches order $2k-1$ where ENO reaches order k. Near a discontinuity the soft decision becomes practically hard, and WENO behaves like ENO.

4.4 Remarks on Accuracy: The Role of ε and WENO-Z

At a **critical point**, where $u'(x_i) = 0$, the leading term of (14) vanishes. Then the β_k are all $\mathcal{O}(\Delta x^4)$ but they differ at relative order $\mathcal{O}(\Delta x)$, so $\omega_k - d_k = \mathcal{O}(\Delta x)$ and the accuracy can drop to third order there (Henrick, Aslam and Powers, 2005). In practice $\varepsilon = 10^{-6}$ hides this on moderate grids: once $\beta_k \ll \varepsilon$, all $\alpha_k \approx d_k/\varepsilon^2$ and the weights are simply the optimal ones. This is also why ε should not be chosen too small.

A popular improvement is **WENO-Z** (Borges, Carmona, Costa and Don, 2008), which uses a global indicator $\tau_5 = |\beta_0 - \beta_2|$ and

$$\alpha_k^Z = d_k\left[1 + \left(\frac{\tau_5}{\varepsilon + \beta_k}\right)^2\right]. \tag{16}$$

In smooth regions $\tau_5 \ll \beta_k$, so $\alpha_k^Z \approx d_k$ and the weights are much closer to optimal, including at critical points. Near a discontinuity $\tau_5 = \mathcal{O}(1)$ and the behaviour is as before. WENO-Z is less dissipative than the classical (Jiang-Shu) WENO at almost no extra cost.

5 The Complete WENO5 Scheme

Algorithm: Fifth-Order WENO Finite Volume Scheme (WENO5)

For each Runge-Kutta stage, given the cell averages $\bar{u}_j$:

1. Fill **three ghost cells** on each side according to the boundary condition (Lecture 11). The stencil for $u^R_{i+1/2}$ reaches $\bar{u}_{i+3}$.
2. For every interface $x_{i+1/2}$:
   (a) compute $u^{(k)}_{i+1/2}$, β_k and ω_k from $\bar{u}_{i-2}, \dots, \bar{u}_{i+2}$, and set $u^L_{i+1/2} = \sum_k \omega_k u^{(k)}_{i+1/2}$;
   (b) compute $u^R_{i+1/2}$ in the same way from the mirrored data $\bar{u}_{i+3}, \bar{u}_{i+2}, \bar{u}_{i+1}, \bar{u}_i, \bar{u}_{i-1}$.
3. Evaluate the numerical flux $F_{i+1/2} = \mathcal{F}(u^L_{i+1/2}, u^R_{i+1/2})$ (Godunov, Rusanov, ...).
4. Form $\mathcal{L}_i(\bar{u}) = -(F_{i+1/2} - F_{i-1/2})/\Delta x$.

Advance in time with **SSP-RK3** (Lecture 12).

For linear advection $u_t + a u_x = 0$ with $a > 0$, the upwind flux is $F_{i+1/2} = a\, u^L_{i+1/2}$, and only the left states are needed.

Time step. WENO is not TVD, so there is no Harten-type proof that fixes the time step. As a guide, the linear scheme (weights d_k) with upwind flux and SSP-RK3 is linearly stable for $\nu = a\Delta t/\Delta x \lesssim 1.43$ (Exercise E6). For shock problems a CFL number of 0.4 to 0.6 is the usual choice, with $\nu = \max_i |f'(\bar{u}_i)|\, \Delta t/\Delta x$.

Order in time. SSP-RK3 is third order, so with $\Delta t = \nu \Delta x$ the time error $\mathcal{O}(\Delta t^3)$ eventually dominates the spatial error $\mathcal{O}(\Delta x^5)$. For most practical problems the spatial error is the larger one on realistic grids, but to *measure* fifth order in a convergence study one must reduce Δt faster, for example $\Delta t \propto \Delta x^{5/3}$, so that $\Delta t^3 \propto \Delta x^5$.

Systems (preview). For systems such as the Euler equations, reconstructing each conservative variable separately can produce oscillations, because different waves interact inside the stencil. The robust approach is to reconstruct in **characteristic variables**, using the eigenvectors of the flux Jacobian. This is the subject of Module 5.

6 Numerical Results

6.1 Linear Advection

Figure 2 repeats the Gaussian plus square wave test of Lecture 12, now with SSP-RK3 for all schemes. The heights of the Gaussian after one period (the exact cell-average maximum is 0.9975) are:

	MUSCL-minmod	MUSCL-MC	ENO3	WENO5
Gaussian peak	0.838	0.953	0.987	0.997

The MUSCL schemes clip the peak; ENO3 and WENO5 keep it. At the square wave, WENO5 is as sharp as MC. WENO5 is not exactly TVD: its minimum value is -3.7×10^{-4}, a tiny undershoot next to the square wave, but it is far below anything visible. This is "essentially non-oscillatory" in action.

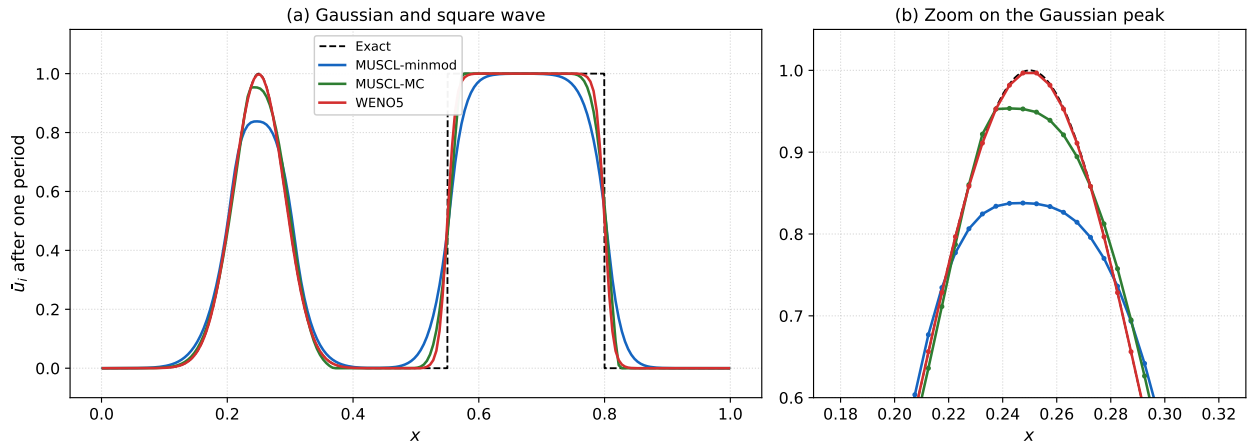


Figure 2: $u_t + u_x = 0$ on a periodic domain, $N = 200$, $\nu = 0.4$, SSP-RK3, after one period. (a) Full solution. (b) Zoom on the Gaussian: minmod and MC clip the peak, WENO5 follows the exact solution.

6.2 Convergence

We advect $u_0(x) = \sin(2\pi x)$ for one period. For ENO3 and WENO5 we take $\Delta t = 0.4\,\Delta x\,(20\,\Delta x)^{2/3}$, so that the time error does not hide the spatial order; for MUSCL $\Delta t = 0.4\,\Delta x$.

	L^1 **error**			L^∞ **error**		
Scheme	$N = 320$	$N = 640$	**order**	$N = 320$	$N = 640$	**order**
MUSCL-minmod	1.03×10^{-3}	2.76×10^{-4}	1.90	7.29×10^{-3}	2.94×10^{-3}	1.31
MUSCL-MC	2.63×10^{-4}	6.66×10^{-5}	1.98	2.31×10^{-3}	9.45×10^{-4}	1.29
ENO3	2.52×10^{-6}	3.15×10^{-7}	3.00	4.18×10^{-6}	5.17×10^{-7}	3.01
WENO5	1.67×10^{-9}	5.24×10^{-11}	5.00	3.05×10^{-9}	8.91×10^{-11}	5.10

Two things stand out. First, ENO3 and WENO5 have the *same* order in L^1 and L^∞: there is no loss of accuracy at the extrema of the sine wave, unlike the TVD schemes. Second, the gap in size is enormous: on 640 cells WENO5 is more than six orders of magnitude more accurate than MUSCL-MC, using the same five-cell neighbourhood.

6.3 Burgers' Equation

Figure 3 shows Burgers' equation with $u_0 = 0.5 + \sin(2\pi x)$ and the Godunov flux, at $t = 0.25$, after the shock has formed (the breaking time is $1/(2\pi) \approx 0.16$). On only 50 cells all schemes capture the shock in one or two cells without oscillations. WENO5 places the cell values closest to the reference on both sides of the shock, and it follows the smooth part of the solution without visible error.

Insight: Cost versus Accuracy

Per time step, WENO5 costs a few times more than MUSCL: three candidate values, three indicators and three weights per interface, and three Runge-Kutta stages. But to reach a given accuracy on a smooth solution it needs far fewer cells. For problems with both smooth structure and shocks, such as turbulence, acoustics or shock-vortex interaction, this trade-off strongly favours WENO, and WENO5 with SSP-RK3 has become a standard tool in compressible flow simulation.

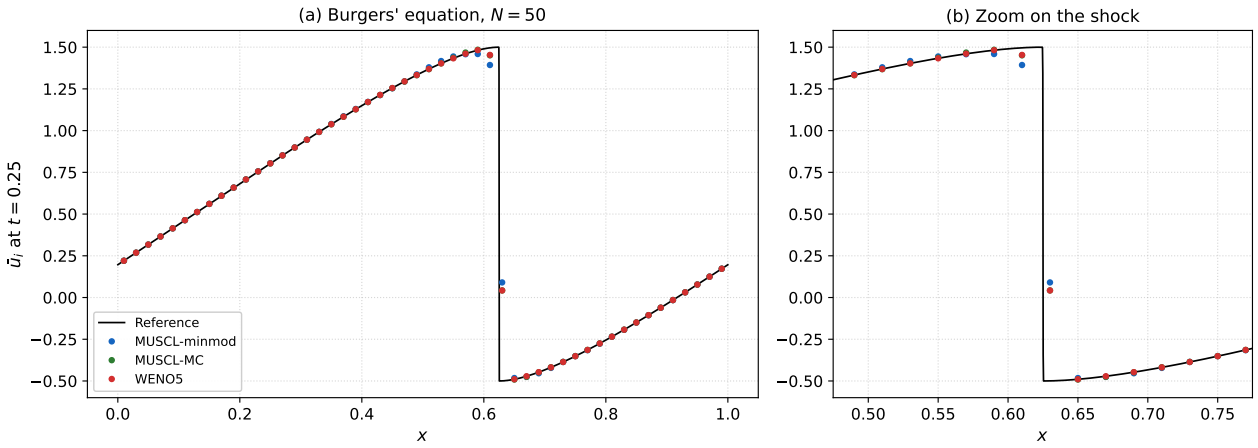


Figure 3: Burgers' equation, $u_0 = 0.5 + \sin(2\pi x)$, periodic, $t = 0.25$, $N = 50$, Godunov flux, SSP-RK3. The reference solution is WENO5 on 4000 cells. (b) Zoom on the shock.

7 Summary

Concept	Statement	Significance
Reconstruction	Polynomial p with the right cell averages on a stencil of k cells	Order k at the interface; conservative
Third-order values	$u_{i+1/2}^{(k)} = \bar{u}_i + \frac{1}{2}b_k + \frac{1}{6}c_k$ on S_0, S_1, S_2	Building blocks of ENO3 and WENO5
ENO	Grow the stencil towards the smaller difference	No clipping; but not TVD, wastes cells, non-smooth
Linear weights	$d = (\frac{1}{10}, \frac{6}{10}, \frac{3}{10})$	Combine three third-order values into a fifth-order value
Smoothness indicator	$\beta_k = b_k^2 + \frac{13}{3}c_k^2$	$\mathcal{O}(\Delta x^2)$ if smooth, $\mathcal{O}(1)$ across a jump
WENO weights	$\omega_k \propto d_k/(\varepsilon + \beta_k)^2$	$\omega \approx d$ when smooth; $\omega_k \approx 0$ on a bad stencil
WENO5	WENO reconstruction + monotone flux + SSP-RK3	Fifth order in L^1 and L^∞, sharp shocks

8 Exercises

Exercises for Lecture 13

E1. Reconstruction Formulas:

(a) Using (5) with $m = -2, -1, 0$, find a_0, b_0, c_0 and show that $u^{(0)}_{i+1/2} = \frac{1}{3}\bar{u}_{i-2} - \frac{7}{6}\bar{u}_{i-1} + \frac{11}{6}\bar{u}_i$. Repeat for S_2.

(b) Verify that $\frac{1}{10}u^{(0)}_{i+1/2} + \frac{6}{10}u^{(1)}_{i+1/2} + \frac{3}{10}u^{(2)}_{i+1/2}$ equals the right-hand side of (8).

(c) Check that all formulas reproduce constants and linear functions exactly: take $\bar{u}_j = \alpha + \beta j$ (these are the averages of a linear function) and compute $u^{(k)}_{i+1/2}$.

E2. Smoothness Indicators:

(a) Show that the cell averages of a smooth u satisfy $\bar{u}_{i+m} = u + m\Delta x\, u' + (m^2 + \frac{1}{12})\frac{\Delta x^2}{2}u'' + \mathcal{O}(\Delta x^3)$, with u and its derivatives at x_i.

(b) Deduce the expansions (7) of b_k and c_k for $k = 0, 1, 2$, and hence (14).

(c) Starting from (12), derive the expressions for β_0 and β_2 in (13).

E3. ENO and WENO by Hand: The data $\bar{u}_{i-2}, \dots, \bar{u}_{i+2} = (0, 0, 0, 1, 2)$ have a kink at cell i.

(a) Find the ENO stencil and the ENO value of $u^L_{i+1/2}$.

(b) Compute $u^{(k)}_{i+1/2}$, β_k and ω_k ($\varepsilon = 10^{-6}$) for $k = 0, 1, 2$, and the WENO value of $u^L_{i+1/2}$.

(c) Compare with the value from the linear weights d_k. Which one creates an overshoot or undershoot?

E4. ENO versus Minmod at an Extremum: Take $\bar{u}_{i-1} = 0$, $\bar{u}_i = 1$, $\bar{u}_{i+1} = 0.8$ (a discrete maximum at i).

(a) Compute the minmod slope and the second-order ENO slope in cell i, and the resulting $u^L_{i+1/2}$.

(b) Explain, using this example, why ENO does not clip extrema and why it is not TVD.

E5. The Right State: Write out explicitly the three candidate values and the three smoothness indicators for $u^R_{i+1/2}$ in terms of $\bar{u}_{i-1}, \dots, \bar{u}_{i+3}$. What are the linear weights for the stencils $\{i+1, i+2, i+3\}$, $\{i, i+1, i+2\}$ and $\{i-1, i, i+1\}$?

E6. Linear Stability of WENO5 with SSP-RK3: For $u_t + u_x = 0$ with the linear weights, show that the Fourier symbol of the semi-discrete operator is $\Delta x\, \hat{\mathcal{L}}(\theta) = -\big(1 - e^{-\mathrm{i}\theta}\big)\, s(\theta)$ with $s(\theta) = \frac{1}{60}\big(2e^{-2\mathrm{i}\theta} - 13e^{-\mathrm{i}\theta} + 47 + 27e^{\mathrm{i}\theta} - 3e^{2\mathrm{i}\theta}\big)$. Using the amplification factor $G = 1 + z + z^2/2 + z^3/6$, $z = \nu\, \Delta x\, \hat{\mathcal{L}}(\theta)$, of SSP-RK3, plot $\max_\theta |G|$ against ν and confirm the limit $\nu \approx 1.43$.

E7. Computational Exercise: Run `advection_weno5.jl` from the course website.

(a) Compare WENO5, ENO3 and MUSCL-minmod after 10 periods. Which scheme keeps the Gaussian best, and which keeps the square wave best?

(b) Replace $\varepsilon = 10^{-6}$ by 10^{-40} and by 10^{-2}. What changes, and why?

(c) Implement the WENO-Z weights and compare with the classical weights.

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