MA4110/MA5020: Computational Methods for Fluid Flow

Lecture 7: Von Neumann Stability Analysis and Classical Schemes for Advection

Instructor: Biswarup Biswas

1 Introduction

In Lecture 6, we established the foundations of the Finite Difference Method (FDM) and proved that both the Forward-Time Central-Space (FTCS) and First-Order Upwind (FOU) schemes are formally consistent with the linear advection equation $u_t + au_x = 0$. However, we noted a critical caveat: when executed on a computer, the FTCS scheme rapidly explodes due to catastrophic growth of numerical round-off errors.

This brings us to the second pillar of the Lax Equivalence Theorem: **Numerical Stability**. In this lecture, we introduce the standard quantitative tool for analyzing the stability of linear finite difference schemes: **Von Neumann (Fourier) Stability Analysis**. We will:

1. Formulate the Fourier decomposition of discrete error modes.
2. Define the **amplification factor** $G(\xi)$ and the Von Neumann stability criterion $|G(\xi)| \leq 1$.
3. Prove why FTCS is unconditionally unstable for hyperbolic advection.
4. Derive the conditional stability of the First-Order Upwind (FOU) scheme.
5. Introduce and analyze the classical **Lax-Friedrichs** and **Lax-Wendroff** schemes.

2 The Concept of Numerical Stability

Let u_j^n be the exact algebraic solution to a finite difference equation, and let $\tilde{u}_j^n$ be the actual computed floating-point numerical solution. The **numerical error** at grid point (x_j, t^n) is:

$$\epsilon_j^n = \tilde{u}_j^n - u_j^n \tag{1}$$

Because the difference equations are linear, the error ϵ_j^n satisfies the exact same difference equation as the solution u_j^n.

Definition: Numerical Stability

A numerical scheme is said to be **stable** if small errors (such as machine round-off or initial perturbations) remain bounded as the calculation proceeds forward in time ($n \to \infty$ at fixed Δt). Conversely, if errors grow exponentially without bound as n increases, the scheme is **unstable**.

3 The Von Neumann Stability Analysis

Introduced by John von Neumann during the Manhattan Project, this method uses **discrete Fourier series** to track the growth or decay of individual spatial harmonic modes as time advances.

3.1 Fourier Representation of Grid Functions

Any spatial grid function ϵ_j^n on a domain with periodic boundary conditions can be represented as a sum of discrete Fourier modes:

$$\epsilon_j^n = \sum_k \hat{\epsilon}_k^n \, e^{\mathrm{i}kx_j} = \sum_k \hat{\epsilon}_k^n \, e^{\mathrm{i}k(j\Delta x)} \tag{2}$$

where k is the spatial wavenumber and $\mathrm{i} = \sqrt{-1}$. Let us define the **dimensionless phase angle** ξ:

$$\xi = k\,\Delta x \in [-\pi, \pi] \tag{3}$$

- $\xi = 0$ ($k = 0$): Represents the infinitely long wave (constant mode).
- $\xi = \pm\pi$ ($k = \pm\pi/\Delta x$, wavelength $\lambda = 2\Delta x$): Represents the **highest frequency mode resolvable on the grid** (the high-frequency saw-tooth mode, alternating $\dots, +1, -1, +1, -1, \dots$).

3.2 Separation of Variables and the Amplification Factor $G(\xi)$

Because the difference equation is linear and has constant coefficients, the spatial Fourier modes evolve independently without interacting. We can therefore analyze a single representative harmonic mode using **separation of variables**:

$$\epsilon_j^n = \hat{\epsilon}^n\, e^{\mathrm{i}j\xi} \tag{4}$$

where $\hat{\epsilon}^n$ denotes the time-dependent amplitude and $e^{\mathrm{i}j\xi}$ captures the spatial oscillation.

At the next time level $t^{n+1} = (n+1)\Delta t$, this mode becomes:

$$\epsilon_j^{n+1} = \hat{\epsilon}^{n+1}\, e^{\mathrm{i}j\xi} \tag{5}$$

When the ansatz (**??**) is substituted into a linear difference equation, the common spatial factor $e^{\mathrm{i}j\xi}$ cancels throughout, directly relating the amplitudes at consecutive time levels:

$$\hat{\epsilon}^{n+1} = G(\xi)\,\hat{\epsilon}^n \tag{6}$$

Here, the complex multiplier $G(\xi)$ is called the **amplification factor**:

$$\boxed{G(\xi) = \frac{\hat{\epsilon}^{n+1}}{\hat{\epsilon}^n}} \tag{7}$$

It measures how much the amplitude of a Fourier mode with phase angle $\xi = k\Delta x$ is magnified (or attenuated) across a single time step Δt.

3.3 The Von Neumann Stability Criterion

Applying the recursive relation (**??**) successively across n time steps yields:

$$\hat{\epsilon}^n = G(\xi)\,\hat{\epsilon}^{n-1} = [G(\xi)]^2\,\hat{\epsilon}^{n-2} = \cdots = [G(\xi)]^n\,\hat{\epsilon}^0 \tag{8}$$

To ensure that error amplitudes do not grow exponentially as the simulation advances ($n \to \infty$), the magnitude of the amplification factor must not exceed unity for all resolvable spatial frequencies:

$$\boxed{|G(\xi)| \le 1 \qquad \forall \xi \in [-\pi, \pi]} \tag{9}$$

4 Stability Analysis of Classical Schemes for Advection

Because all difference schemes considered below are **linear**, the error ϵ_j^n satisfies the exact same algebraic equation as the numerical solution u_j^n. Therefore, substituting the harmonic mode $u_j^n = \hat{u}^n e^{\mathrm{i}j\xi}$ directly into the scheme yields the exact same amplification factor $G(\xi)$ as substituting ϵ_j^n.

We now apply Von Neumann analysis to the 1D linear advection equation:

$$\frac{\partial u}{\partial t} + a\frac{\partial u}{\partial x} = 0, \qquad a > 0 \tag{10}$$

Recall the **CFL (Courant) number**:

$$\nu = \frac{a\,\Delta t}{\Delta x} \tag{11}$$

4.1 Forward Time, Central Space (FTCS)

The FTCS difference equation is:

$$\frac{u_j^{n+1} - u_j^n}{\Delta t} + a\frac{u_{j+1}^n - u_{j-1}^n}{2\Delta x} = 0 \implies u_j^{n+1} = u_j^n - \frac{\nu}{2}\left(u_{j+1}^n - u_{j-1}^n\right) \tag{12}$$

Substituting a Fourier mode $u_j^n = \hat{u}^n e^{ij\xi}$:

$$\begin{aligned} \hat{u}^{n+1} e^{\mathrm{i}j\xi} &= \hat{u}^n e^{\mathrm{i}j\xi} - \frac{\nu}{2}\left(\hat{u}^n e^{\mathrm{i}(j+1)\xi} - \hat{u}^n e^{\mathrm{i}(j-1)\xi}\right) \\ \hat{u}^{n+1} &= \hat{u}^n \left[1 - \frac{\nu}{2}\left(e^{\mathrm{i}\xi} - e^{-\mathrm{i}\xi}\right)\right] \end{aligned} \tag{13}$$

Using Euler's identity $\sin\xi = \frac{e^{\mathrm{i}\xi} - e^{-\mathrm{i}\xi}}{2\mathrm{i}}$:

$$G_{\text{FTCS}}(\xi) = 1 - \mathrm{i}\nu\sin\xi \tag{14}$$

Computing the squared magnitude of the amplification factor:

$$|G_{\text{FTCS}}(\xi)|^2 = 1^2 + (-\nu\sin\xi)^2 = 1 + \nu^2\sin^2\xi \tag{15}$$

Conclusion: FTCS is Unconditionally Unstable

For any non-zero time step ($\nu > 0$), we have:

$$|G_{\text{FTCS}}(\xi)| = \sqrt{1 + \nu^2\sin^2\xi} > 1 \qquad \text{for all } \xi \neq 0, \pm\pi$$

Every interior Fourier mode is amplified at every time step ($|G| > 1$). The high frequencies blow up exponentially. Therefore, **FTCS is unconditionally unstable for hyperbolic advection**.

4.2 First-Order Upwind Scheme (FOU)

For $a > 0$, the backward-in-space upwind scheme is:

$$u_j^{n+1} = u_j^n - \nu\left(u_j^n - u_{j-1}^n\right) = (1-\nu)u_j^n + \nu u_{j-1}^n \tag{16}$$

Substituting $u_j^n = \hat{u}^n e^{ij\xi}$:

$$\begin{aligned} \hat{u}^{n+1} &= \hat{u}^n \left[(1-\nu) + \nu e^{-\mathrm{i}\xi}\right] \\ G_{\text{FOU}}(\xi) &= (1-\nu) + \nu(\cos\xi - \mathrm{i}\sin\xi) = (1 - \nu + \nu\cos\xi) - \mathrm{i}\nu\sin\xi \end{aligned} \tag{17}$$

Computing the squared magnitude $|G_{\text{FOU}}(\xi)|^2$:

$$\begin{aligned} |G_{\text{FOU}}(\xi)|^2 &= (1 - \nu + \nu\cos\xi)^2 + \nu^2\sin^2\xi \\ &= (1-\nu)^2 + 2\nu(1-\nu)\cos\xi + \nu^2\cos^2\xi + \nu^2\sin^2\xi \\ &= 1 - 2\nu + \nu^2 + 2\nu(1-\nu)\cos\xi + \nu^2 \\ &= 1 - 2\nu(1-\nu)(1-\cos\xi) \end{aligned} \tag{18}$$

Using the trigonometric identity $1 - \cos\xi = 2\sin^2(\xi/2)$:

$$\boxed{|G_{\text{FOU}}(\xi)|^2 = 1 - 4\nu(1-\nu)\sin^2\left(\frac{\xi}{2}\right)} \tag{19}$$

For stability, we require $|G(\xi)|^2 \leq 1$, which implies:

$$4\nu(1-\nu)\sin^2\left(\frac{\xi}{2}\right) \geq 0 \quad \text{and} \quad 4\nu(1-\nu)\sin^2\left(\frac{\xi}{2}\right) \leq 1 \tag{20}$$

This holds for all ξ if and only if $\nu(1-\nu) \geq 0$, which yields the celebrated **CFL stability condition**:

$$\boxed{0 \leq \nu = \frac{a\,\Delta t}{\Delta x} \leq 1} \tag{21}$$

4.3 The Lax-Friedrichs Scheme

In 1954, Peter Lax proposed a simple cure for the instability of FTCS: replace the central node value u_j^n in the time derivative by the spatial average of its neighbors, $\frac{u_{j+1}^n + u_{j-1}^n}{2}$:

$$\frac{u_j^{n+1} - \frac{1}{2}(u_{j+1}^n + u_{j-1}^n)}{\Delta t} + a\frac{u_{j+1}^n - u_{j-1}^n}{2\Delta x} = 0 \tag{22}$$

Solving for u_j^{n+1}:

$$\boxed{u_j^{n+1} = \frac{1}{2}(u_{j+1}^n + u_{j-1}^n) - \frac{\nu}{2}(u_{j+1}^n - u_{j-1}^n) = \frac{1-\nu}{2}u_{j+1}^n + \frac{1+\nu}{2}u_{j-1}^n} \tag{23}$$

4.3.1 Stability of Lax-Friedrichs

Substituting $u_j^n = \hat{u}^n e^{\mathrm{i}j\xi}$ into equation (**??**):

$$G_{\mathrm{LxF}}(\xi) = \frac{e^{\mathrm{i}\xi} + e^{-\mathrm{i}\xi}}{2} - \frac{\nu}{2}(e^{\mathrm{i}\xi} - e^{-\mathrm{i}\xi}) = \cos\xi - \mathrm{i}\nu\sin\xi \tag{24}$$

Computing the magnitude squared:

$$|G_{\mathrm{LxF}}(\xi)|^2 = \cos^2\xi + \nu^2\sin^2\xi = 1 - \sin^2\xi + \nu^2\sin^2\xi = 1 - (1-\nu^2)\sin^2\xi \tag{25}$$

For $|G_{\mathrm{LxF}}(\xi)| \le 1$ for all ξ, we require $1 - \nu^2 \ge 0 \implies \nu^2 \le 1$. Thus, the Lax-Friedrichs scheme is **conditionally stable** under:

$$\boxed{|\nu| = \frac{|a|\Delta t}{\Delta x} \le 1} \tag{26}$$

4.3.2 Modified Equation and Dissipation of Lax-Friedrichs

Notice that equation (**??**) can be rewritten as FTCS plus an artificial diffusion term:

$$\frac{u_j^{n+1} - u_j^n}{\Delta t} + a\frac{u_{j+1}^n - u_{j-1}^n}{2\Delta x} = \frac{\Delta x^2}{2\Delta t}\left(\frac{u_{j+1}^n - 2u_j^n + u_{j-1}^n}{\Delta x^2}\right) \tag{27}$$

The modified equation is:

$$\frac{\partial u}{\partial t} + a\frac{\partial u}{\partial x} = \underbrace{\left(\frac{\Delta x^2}{2\Delta t} - \frac{a^2\Delta t}{2}\right)}_{\alpha_{\mathrm{num}} = \frac{\Delta x^2}{2\Delta t}(1-\nu^2)}\frac{\partial^2 u}{\partial x^2} \tag{28}$$

Lax-Friedrichs adds a large amount of artificial numerical dissipation $\mathcal{O}(\Delta x^2/\Delta t)$, which stabilizes the scheme but smears out sharp gradients.

4.4 The Lax-Wendroff Scheme (Second-Order)

The First-Order Upwind and Lax-Friedrichs schemes are only first-order accurate ($\mathcal{O}(\Delta t, \Delta x)$). In 1960, Peter Lax and Burton Wendroff developed a **second-order accurate in space and time** scheme ($\mathcal{O}(\Delta t^2, \Delta x^2)$). The key idea is simple: expand u in time via Taylor series, and use the PDE $u_t = -au_x$ to replace time derivatives with spatial derivatives.

4.4.1 Derivation

Expand $u(x_j, t^{n+1})$ in time via Taylor series up to second order:

$$u(x_j, t^{n+1}) = u(x_j, t^n) + \Delta t\, \frac{\partial u}{\partial t}(x_j, t^n) + \frac{\Delta t^2}{2}\, \frac{\partial^2 u}{\partial t^2}(x_j, t^n) + \mathcal{O}(\Delta t^3) \tag{29}$$

Using the advection PDE $u_t = -au_x$, we replace time derivatives with spatial derivatives:

$$\frac{\partial u}{\partial t} = -a\frac{\partial u}{\partial x} \tag{30}$$

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial}{\partial t}\left(-a\frac{\partial u}{\partial x}\right) = -a\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial t}\right) = -a\frac{\partial}{\partial x}\left(-a\frac{\partial u}{\partial x}\right) = a^2\frac{\partial^2 u}{\partial x^2} \tag{31}$$

Substituting these expressions into equation (**??**):

$$u(x_j, t^{n+1}) = u(x_j, t^n) - a\Delta t\, \frac{\partial u}{\partial x} + \frac{a^2 \Delta t^2}{2}\, \frac{\partial^2 u}{\partial x^2} + \mathcal{O}(\Delta t^3) \tag{32}$$

Now, replace spatial derivatives with standard second-order central difference formulas:

$$\boxed{u_j^{n+1} = u_j^n - \frac{\nu}{2}(u_{j+1}^n - u_{j-1}^n) + \frac{\nu^2}{2}(u_{j+1}^n - 2u_j^n + u_{j-1}^n)} \tag{33}$$

4.4.2 Stability Analysis of Lax-Wendroff

Substituting the Fourier mode $u_j^n = \hat{u}^n e^{\mathrm{i}j\xi}$:

$$\begin{aligned} G_{\mathrm{LW}}(\xi) &= 1 - \frac{\nu}{2}(e^{\mathrm{i}\xi} - e^{-\mathrm{i}\xi}) + \frac{\nu^2}{2}(e^{\mathrm{i}\xi} - 2 + e^{-\mathrm{i}\xi}) \\ &= 1 - \mathrm{i}\nu \sin\xi + \nu^2(\cos\xi - 1) = (1 - 2\nu^2 \sin^2(\xi/2)) - \mathrm{i}\nu \sin\xi \end{aligned} \tag{34}$$

Computing the squared magnitude:

$$\begin{aligned} |G_{\mathrm{LW}}(\xi)|^2 &= \left(1 - 2\nu^2 \sin^2(\xi/2)\right)^2 + \nu^2 \sin^2\xi \\ &= 1 - 4\nu^2 \sin^2(\xi/2) + 4\nu^4 \sin^4(\xi/2) + 4\nu^2 \sin^2(\xi/2)\cos^2(\xi/2) \\ &= 1 - 4\nu^2 \sin^2(\xi/2)\left[1 - \cos^2(\xi/2) - \nu^2 \sin^2(\xi/2)\right] \\ &= 1 - 4\nu^2 \sin^2(\xi/2)\left[\sin^2(\xi/2)(1 - \nu^2)\right] \\ &= \boxed{1 - 4\nu^2(1-\nu^2)\sin^4\left(\frac{\xi}{2}\right)} \end{aligned} \tag{35}$$

For $|G_{\mathrm{LW}}(\xi)| \leq 1$ for all ξ, we require $4\nu^2(1 - \nu^2) \geq 0 \implies \nu^2 \leq 1$. Thus, Lax-Wendroff is **second-order accurate and stable** under:

$$\boxed{|\nu| = \frac{|a|\Delta t}{\Delta x} \leq 1} \tag{36}$$

5 Summary of Classical Advection Schemes

Scheme	**Order**	**Amplification Factor $G(\xi)$**	**Stability Condition**
FTCS	$\mathcal{O}(\Delta t, \Delta x^2)$	$1 - \mathrm{i}\nu \sin\xi$	**Unconditionally Unstable**
Upwind (FOU)	$\mathcal{O}(\Delta t, \Delta x)$	$(1 - \nu + \nu\cos\xi) - \mathrm{i}\nu \sin\xi$	$0 \leq \nu \leq 1$ $(a > 0)$
Lax-Friedrichs	$\mathcal{O}(\Delta t, \Delta x^2/\Delta t)$	$\cos\xi - \mathrm{i}\nu \sin\xi$	$|\nu| \leq 1$
Lax-Wendroff	$\mathcal{O}(\Delta t^2, \Delta x^2)$	$1 - 2\nu^2 \sin^2(\xi/2) - \mathrm{i}\nu \sin\xi$	$|\nu| \leq 1$

6 Exercises

Exercises for Lecture 7

E1. Diffusion Equation Stability: Consider the 1D heat equation $u_t = \alpha u_{xx}$ with $\alpha > 0$, discretized using Forward-Time Central-Space (FTCS):

$$\frac{u_j^{n+1} - u_j^n}{\Delta t} = \alpha \frac{u_{j+1}^n - 2u_j^n + u_{j-1}^n}{\Delta x^2}$$

(a) Define the diffusion number $r = \frac{\alpha \Delta t}{\Delta x^2}$ and express the scheme in update form.

(b) Apply Von Neumann stability analysis to find the amplification factor $G(\xi)$ as a function of r and ξ.

(c) Prove that the scheme is stable if and only if $r \leq \frac{1}{2}$.

E2. Dissipation and Dispersion Errors: For the Lax-Wendroff scheme with $\nu = 0.5$, compute the magnitude $|G(\xi)|$ at the wave modes $\xi = \pi/4, \pi/2, \pi$. Which wave modes suffer the most numerical damping?

E3. Beam-Warming Scheme (Upwind Second-Order): By expanding $u(x_j, t^{n+1})$ in time and replacing spatial derivatives with second-order *backward* differences, derive the explicit update formula for the second-order upwind **Beam-Warming scheme**:

$$u_j^{n+1} = u_j^n - \frac{\nu}{2}(3u_j^n - 4u_{j-1}^n + u_{j-2}^n) + \frac{\nu^2}{2}(u_j^n - 2u_{j-1}^n + u_{j-2}^n)$$

Show that its CFL stability condition is $0 \leq \nu \leq 2$.