MA4110/MA5020: Computational Methods for Fluid Flow

Lecture 5: Method of Characteristics: Non-linear Problems & Weak Solutions

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1 Introduction

In Lecture 4, we applied the method of characteristics to the linear advection equation. Because the wave speed a was constant, all characteristic curves were parallel straight lines, and initial profiles translated through space without deformation.

In this lecture, we investigate the **inviscid Burgers' equation**, a prototype non-linear scalar conservation law where the propagation speed depends directly on the local state u. This non-linearity yields two fundamental phenomena:

- **Wave Steepening and Compression:** Regions where larger values trail smaller values cause characteristics to converge and intersect, resulting in the breakdown of smooth classical solutions and the generation of **shock waves**.
- **Wave Spreading and Expansion:** Regions where smaller values trail larger values cause characteristics to diverge, resulting in continuous **rarefaction waves**.

We will develop the analytical framework for characteristics, establish when and why classical solutions fail, formulate the integral balance governing **weak solutions**, and apply the **Rankine-Hugoniot** and **Lax entropy conditions** to resolve both general non-linear waves and the canonical **Riemann problem**.

2 The Inviscid Burgers' Equation

The one-dimensional inviscid Burgers' equation is given by:

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = 0, \qquad x \in \mathbb{R},\ t > 0 \tag{1}$$

with general initial data:

$$u(x,0) = u_0(x) \tag{2}$$

Equation (1) is a **quasilinear** first-order PDE. The effective advection speed is $a(u) = u$, meaning that crests (higher u) travel faster than troughs (lower u).

3 The Method of Characteristics for Non-Linear Equations

3.1 Characteristic Differential Equations

Consider a curve $x = x(t)$ in the space-time (x,t) plane. The total rate of change of u along this path is:

$$\frac{du}{dt} = \frac{\partial u}{\partial t} + \frac{dx}{dt}\frac{\partial u}{\partial x} \tag{3}$$

Comparing this expansion directly with equation (1), we obtain the characteristic equations:

$$\frac{dx}{dt} = u(x(t),t) \tag{4}$$

$$\frac{du}{dt} = 0 \tag{5}$$

3.2 Integration of Characteristics

From equation (5), u is **strictly constant** along each characteristic path. For a path originating from $x = x_0$ at $t = 0$:

$$u(x(t), t) = u(x_0, 0) = u_0(x_0) \tag{6}$$

Because u remains constant at the value $u_0(x_0)$, equation (4) integrates to a straight line:

$$\boxed{x(t) = x_0 + u_0(x_0)\, t} \tag{7}$$

Key Geometric Property

Every characteristic curve of the inviscid Burgers' equation is a straight line in the (x, t) plane. However, unlike the linear advection equation where all lines are parallel, the slope $\frac{dt}{dx} = \frac{1}{u_0(x_0)}$ is determined by the initial value $u_0(x_0)$ at the foot of that line.

4 Slope Evolution: Wave Steepening vs. Expansion

4.1 Evolution of the Spatial Derivative

To analyze profile deformation, consider the spatial derivative $\frac{\partial u}{\partial x}(x, t)$ along a characteristic starting at x_0. Using the chain rule:

$$\frac{\partial u}{\partial x} = u_0'(x_0) \frac{\partial x_0}{\partial x} \tag{8}$$

Differentiating the characteristic relation $x = x_0 + u_0(x_0)t$ with respect to x at fixed time t:

$$1 = \frac{\partial x_0}{\partial x} + u_0'(x_0)\, t\, \frac{\partial x_0}{\partial x} = \left[1 + u_0'(x_0)\, t\right] \frac{\partial x_0}{\partial x} \implies \frac{\partial x_0}{\partial x} = \frac{1}{1 + u_0'(x_0)\, t} \tag{9}$$

Substituting this back into equation (8) yields the exact formula for the evolution of the spatial slope:

$$\boxed{\frac{\partial u}{\partial x}(x, t) = \frac{u_0'(x_0)}{1 + u_0'(x_0)\, t}} \tag{10}$$

4.2 Two Competing Behaviors: Expansion vs. Compression

Equation (10) distinguishes two distinct physical behaviors depending on the sign of $u_0'(x_0)$:

1. **Expansion ($u_0'(x_0) > 0$):** The initial profile is increasing. The denominator $1 + u_0'(x_0)t > 1$ grows monotonically with t. Consequently:

$$\left|\frac{\partial u}{\partial x}\right| \to 0 \quad \text{as} \quad t \to \infty$$

   The wave spreads out smoothly. Characteristics diverge from one another, and no intersections can ever occur.

2. **Compression and Wave Steepening ($u_0'(x_0) < 0$):** The initial profile is decreasing. Faster-moving states behind overtake slower-moving states ahead. As t increases, the denominator $1 + u_0'(x_0)t$ decreases toward zero, causing the slope $|\frac{\partial u}{\partial x}|$ to grow without bound.

4.3 Breaking Time and Classical Breakdown

At the critical time when the denominator in equation (10) reaches zero, the spatial derivative blows up ($\frac{\partial u}{\partial x} \to -\infty$). The earliest time at which a vertical tangent develops in the domain is called the **breaking time** t_s:

$$\boxed{t_s = \min_{x_0,\, u_0'(x_0)<0} \left(-\frac{1}{u_0'(x_0)} \right) = \frac{-1}{\min_{x_0} u_0'(x_0)}} \tag{11}$$

For $t > t_s$, characteristics intersect, the mapping $x_0 \mapsto x$ is no longer invertible, and classical C^1 solutions cease to exist.

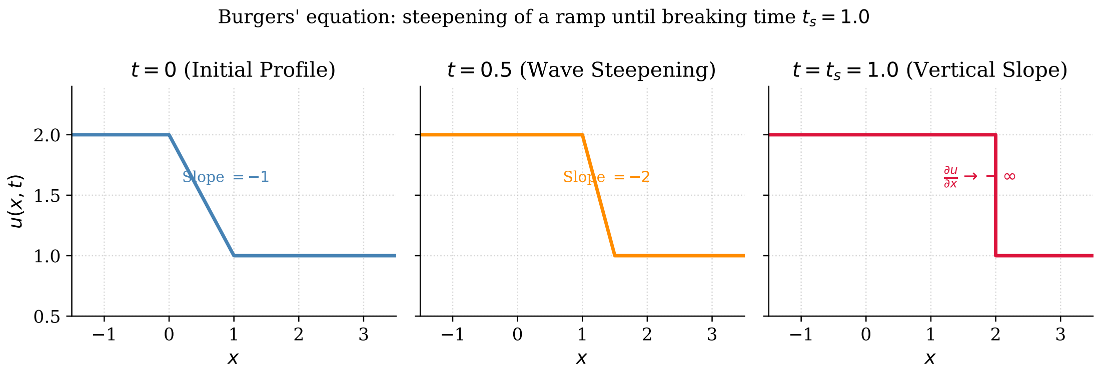

Figure 1: Steepening of a compressive wave in Burgers' equation until the breaking time $t_s = 1.0$, where $\frac{\partial u}{\partial x} \to -\infty$.

5 Weak Solutions and the General Rankine-Hugoniot Condition

5.1 Integral Form of Conservation Laws

To extend solutions past the breaking time $t > t_s$, we rewrite the PDE in **conservation form**:

$$\frac{\partial u}{\partial t} + \frac{\partial f(u)}{\partial x} = 0, \qquad f(u) = \frac{u^2}{2} \tag{12}$$

Integrating over any spatial interval $[x_1, x_2]$ and time interval $[t_1, t_2]$ produces the **integral conservation law**:

$$\int_{x_1}^{x_2} u(x, t_2)\, dx - \int_{x_1}^{x_2} u(x, t_1)\, dx = -\int_{t_1}^{t_2} \left[f\big(u(x_2, t)\big) - f\big(u(x_1, t)\big) \right] dt \tag{13}$$

A bounded, measurable function $u(x, t)$ satisfying equation (13) for all control volumes is termed a **weak solution**. The integral form requires no spatial or temporal derivatives of u, making it valid across discontinuities.

5.2 The Rankine-Hugoniot Jump Condition (General Discontinuity)

Suppose a solution $u(x, t)$ is smooth on either side of a smooth discontinuity curve $x = x_s(t)$, with instantaneous propagation speed:

$$\dot{x}_s(t) = \frac{dx_s}{dt} = s(t) \tag{14}$$

Let the one-sided limits of u immediately to the left and right of the curve at time t be:

$$u^-(t) = \lim_{x \to x_s(t)^-} u(x,t), \qquad u^+(t) = \lim_{x \to x_s(t)^+} u(x,t) \tag{15}$$

Applying the integral balance (13) across an infinitesimal control volume $[x_s(t) - \epsilon, x_s(t) + \epsilon]$ yields the **Rankine-Hugoniot jump condition**:

$$\boxed{s(t) = \frac{f(u^+(t)) - f(u^-(t))}{u^+(t) - u^-(t)}} \tag{16}$$

For the flux function $f(u) = u^2/2$ of Burgers' equation:

$$s(t) = \frac{\frac{1}{2}(u^+)^2 - \frac{1}{2}(u^-)^2}{u^+ - u^-} = \frac{(u^+ - u^-)(u^+ + u^-)}{2(u^+ - u^-)} = \boxed{\frac{u^-(t) + u^+(t)}{2}} \tag{17}$$

General vs. Constant Discontinuities

Equation (17) holds locally at every instant t. If the background states $u^-(t)$ and $u^+(t)$ vary as the shock traverses non-uniform regions of flow, the shock speed $s(t)$ changes continuously in time, producing a **curved shock trajectory** $x_s(t) = \int s(t)\,dt$.

6 Admissibility: The Lax Entropy Condition

6.1 Non-Uniqueness of Weak Solutions

The Rankine-Hugoniot condition (16) is a necessary consequence of conservation, but it is **not sufficient** to guarantee a physically meaningful solution. Multiple weak solutions can satisfy the integral balance for identical initial data.

6.2 The Lax Entropy Criterion

For a scalar conservation law with convex flux ($f''(u) > 0$), a discontinuity moving along $x = x_s(t)$ is physically admissible if and only if characteristics from both sides impinge upon and terminate at the discontinuity. This is formalised by the **Lax entropy condition**:

$$\boxed{f'(u^-(t)) > \dot{x}_s(t) > f'(u^+(t))} \tag{18}$$

For Burgers' equation, where $f'(u) = u$ and $\dot{x}_s(t) = s(t)$:

$$\boxed{u^-(t) > s(t) > u^+(t)} \tag{19}$$

6.3 Geometric and Physical Meaning

- **Compressive Shock ($u^- > u^+$):** Left characteristics move at speed $u^- > s$ (catching up with the shock), while right characteristics move at speed $u^+ < s$ (being overtaken by the shock). Information flows **into** the shock from both sides and is dissipated. This shock is physically valid.

- **Expansion Shock ($u^- < u^+$):** If one constructs a discontinuity between $u^- < u^+$, the RH speed is still $s = (u^- + u^+)/2$. However, $u^- < s < u^+$, meaning characteristics point **away from** the discontinuity. Information would have to be spontaneously created at the jump, violating causality. Such discontinuities are non-physical and rejected by the Lax condition.

7 The Riemann Problem for Burgers' Equation

7.1 Definition

The canonical building block for understanding non-linear wave interactions is the **Riemann problem**: an initial value problem with piecewise-constant initial data containing a single step jump at $x = 0$:

$$u(x,0) = \begin{cases} u_L, & x < 0 \\ u_R, & x > 0 \end{cases} \tag{20}$$

where u_L and u_R are constants.

7.2 Case 1: The Shock Solution ($u_L > u_R$)

When $u_L > u_R$, the left state is faster than the right state. Characteristics collide for $t > 0$. Because the states on either side are constant, the Rankine-Hugoniot condition (17) yields a constant shock speed:

$$s = \frac{u_L + u_R}{2} \tag{21}$$

The shock trajectory is the straight line $x_s(t) = st$. The Lax condition $u_L > s > u_R$ is satisfied.

Constructing $u(x,t)$ via characteristic tracing:

- For $x < x_s(t) = st$: tracing back along characteristics gives $x_0 = x - u_L t < st - u_L t = (s - u_L)t < 0$ (since $s < u_L$). Thus $x_0 < 0 \implies u(x,t) = u_L$.
- For $x > x_s(t) = st$: tracing back gives $x_0 = x - u_R t > st - u_R t = (s - u_R)t > 0$ (since $s > u_R$). Thus $x_0 > 0 \implies u(x,t) = u_R$.

The exact weak solution is:

$$\boxed{u(x,t) = \begin{cases} u_L, & x < x_s(t) \\ u_R, & x > x_s(t) \end{cases} \qquad \text{where} \quad x_s(t) = s\,t, \quad s = \frac{u_L + u_R}{2}} \tag{22}$$

7.3 Case 2: The Rarefaction Fan ($u_L < u_R$)

When $u_L < u_R$, an expansion shock is ruled out by the Lax entropy condition. Instead, characteristics diverge, creating an expanding gap $u_L t < x < u_R t$ for $t > 0$.

To connect the state u_L continuously to u_R, a family of characteristics fans out from the origin $(x,t) = (0,0)$. A characteristic carrying the constant state u^* travels at speed u^*, following the trajectory:

$$x = u^* t \implies u^* = \frac{x}{t} \tag{23}$$

As u^* varies continuously from u_L to u_R, x/t covers the entire fan region $[u_L, u_R]$. The complete continuous solution is the **centered rarefaction wave**:

$$\boxed{u(x,t) = \begin{cases} u_L, & x < u_L\,t \\ \dfrac{x}{t}, & u_L\,t \le x \le u_R\,t \\ u_R, & x > u_R\,t \end{cases}} \tag{24}$$

At the boundaries $x = u_L t$ and $x = u_R t$, $u(x,t)$ is continuous.

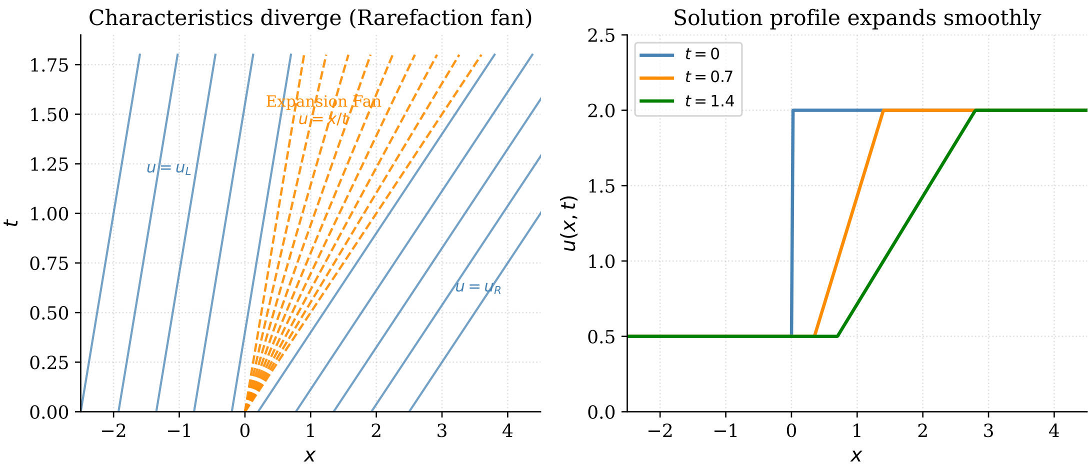

Figure 2: Rarefaction wave ($u_L = 0.5, u_R = 2.0$). Left: characteristic lines fanning out from the origin. Right: self-similar continuous profile spreading linearly with time.

8 Summary of Non-Linear Wave Types

Property	**Shock Wave**	**Rarefaction Wave**
State condition	$u^- > u^+$ (compressive)	$u^- < u^+$ (expansive)
Characteristics	Converge into discontinuity	Diverge, fan out
Mathematical nature	Discontinuous weak solution	Continuous solution
Governing rule	Rankine-Hugoniot: $s = \frac{u^- + u^+}{2}$	Self-similar fan: $u = x/t$
Admissibility	Satisfies Lax: $u^- > s > u^+$	Automatically entropy-satisfying

9 Exercises

Exercises for Lecture 5

E1. General Breaking Time: For Burgers' equation with smooth initial profile $u_0(x) = \frac{1}{1+x^2}$:

(a) Find the region where the profile is compressive ($u_0'(x) < 0$).

(b) Find the location of minimum slope and calculate the breaking time t_s.

(c) Determine the spatial coordinate x_s where the vertical slope first appears at $t = t_s$.

E2. Curved Shock Trajectory: Suppose Burgers' equation has a shock propagating into a non-constant linear background state:

$$u(x,t) = \begin{cases} 2, & x < x_s(t) \\ x, & x > x_s(t) \end{cases}$$

with initial shock position $x_s(0) = 0$.

(a) Set up the differential equation $\frac{dx_s}{dt} = s(t) = \frac{u^- + u^+}{2}$ for the shock position $x_s(t)$.

(b) Solve for $x_s(t)$ explicitly and show that the shock trajectory is curved.

(c) Check whether the Lax entropy condition holds along the shock path for all $t \geq 0$.

E3. Riemann Problem Verification: Solve the Riemann problem for Burgers' equation with $u_L = 4$ and $u_R = -2$.

(a) Identify whether the solution is a shock or a rarefaction.

(b) Compute the wave speed and write down the complete formula for $u(x,t)$.

(c) At what time does the wave cross $x = 5$?

References

[1] Randall J. LeVeque. *Finite Volume Methods for Hyperbolic Problems.* Cambridge University Press, Cambridge, UK, 2002.

[2] Eleuterio F. Toro. *Riemann Solvers and Numerical Methods for Fluid Dynamics: A Practical Introduction.* Springer-Verlag, Berlin, Heidelberg, 3rd edition, 2009.