MA4110/MA5020: Computational Methods for Fluid Flow

Lecture 4: Method of Characteristics : Linear Problems

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1 Introduction

In Lecture 3 we learned that a scalar first-order PDE is hyperbolic and possesses real **characteristic lines** in the (x, t) plane along which information propagates. In this lecture we develop the *method of characteristics*, a technique that converts a PDE into a family of ordinary differential equations by following these characteristic paths.

We apply the method to the simplest hyperbolic PDE: the **linear advection equation**. This gives us an exact analytical solution and builds the geometric intuition we will need when we discretize these equations for numerical simulation.

2 The Linear Advection Equation

Consider a scalar field $u(x, t)$ being transported at a constant speed a:

$$\frac{\partial u}{\partial t} + a\frac{\partial u}{\partial x} = 0, \qquad x \in \mathbb{R},\ t > 0 \tag{1}$$

with initial condition:

$$u(x, 0) = u_0(x) \tag{2}$$

Here a is a **constant** wave speed. If $a > 0$, information moves to the right. If $a < 0$, it moves to the left.

3 Deriving the Method of Characteristics

3.1 Step 1: Introduce a Moving Curve

Suppose we follow a particle moving along a path $x = x(t)$ in the (x, t) plane. The total time derivative of u along this path is:

$$\frac{du}{dt} = \frac{\partial u}{\partial t} + \frac{dx}{dt}\frac{\partial u}{\partial x} \tag{3}$$

3.2 Step 2: Choose the Path to Match the PDE

Compare this with the PDE $\frac{\partial u}{\partial t} + a\frac{\partial u}{\partial x} = 0$. If we choose the path speed to be:

$$\boxed{\frac{dx}{dt} = a} \tag{4}$$

then along this path:

$$\frac{du}{dt} = 0 \tag{5}$$

3.3 Step 3: Solve the Two ODEs

We now have two simple ODEs to solve.
Equation (4): Characteristic path:

$$x(t) = x_0 + at \tag{6}$$

This is a straight line in the (x, t) plane starting at x_0 at $t = 0$, with slope $\frac{1}{a}$ (if we plot t on the vertical axis). These lines are called **characteristic lines**.
Equation (5): Solution along the characteristic:

$$u(x(t), t) = u(x_0, 0) = u_0(x_0) = \text{constant} \tag{7}$$

3.4 Step 4: Express the Solution

Given a point (x, t), the characteristic reaching it originated from $x_0 = x - at$ at $t = 0$. Therefore:

$$\boxed{u(x, t) = u_0(x - at)} \tag{8}$$

Physical Interpretation

The solution $u(x, t) = u_0(x - at)$ is simply the initial profile u_0 translated (shifted) to the right by a distance at at time t. The shape of the profile is **perfectly preserved**. No distortion, no spreading: just pure translation at speed a.

Computational Note

A Julia implementation for evaluating the exact solution of the linear advection equation along characteristics on a spatial grid is provided in **Appendix A**.

3.5 Characteristic Lines in the (x, t) Plane

All characteristic lines for the linear advection equation are **parallel straight lines** with slope $\frac{1}{a}$:

$$x - at = x_0 = \text{const} \tag{9}$$

Figure 1 shows a family of such lines for $a = 1.5$. The key observation is that the value of u is constant along each line and equal to the initial value at x_0.

4 Worked Examples

4.1 Example 1: Step Function Initial Data

Let $a = 1$ and $u_0(x) = \begin{cases} 1 & x \leq 0 \\ 0 & x > 0 \end{cases}$.

By equation (8): $u(x, t) = u_0(x - t) = \begin{cases} 1 & x - t \leq 0 \\ 0 & x - t > 0 \end{cases} = \begin{cases} 1 & x \leq t \\ 0 & x > t \end{cases}$.

The sharp jump (discontinuity) at $x = 0$ simply moves to the right and arrives at $x = t$ at time t.

4.2 Example 2: Sinusoidal Initial Data

Let $a = 2$ and $u_0(x) = \sin(\pi x)$.
Then $u(x, t) = \sin\big(\pi(x - 2t)\big)$.
The sine wave travels to the right at speed $a = 2$.

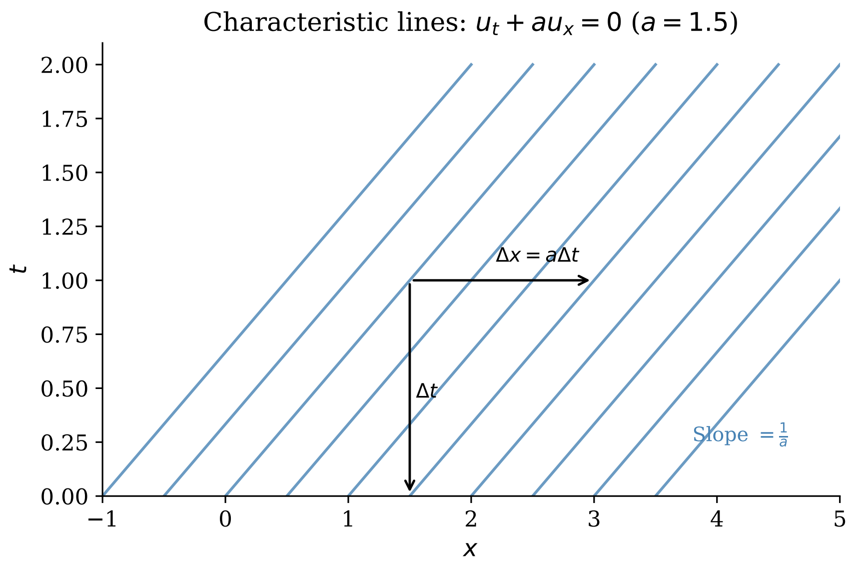

Figure 1: Characteristic lines for $u_t + au_x = 0$ with $a = 1.5$. All lines are parallel with the same slope $\frac{1}{a}$. The value of u is constant along each line.

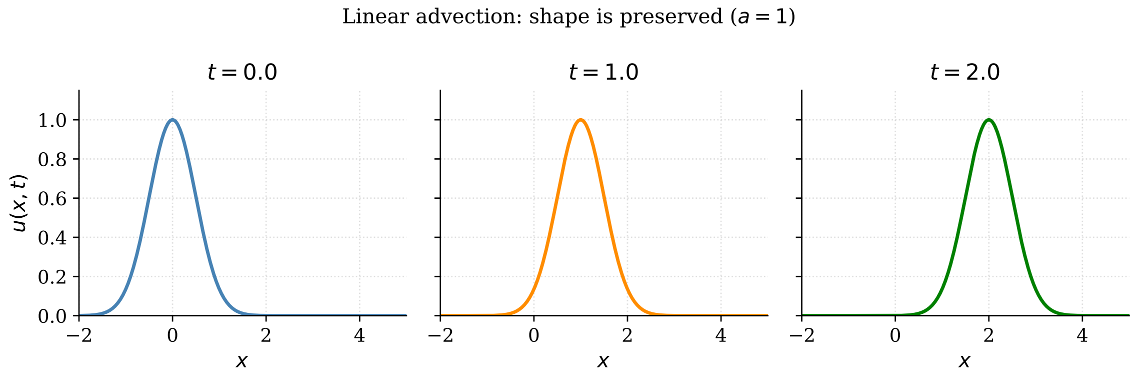

Figure 2: Solution of $u_t + u_x = 0$ at $t = 0, 1, 2$ with a Gaussian initial profile. The profile translates to the right at speed $a = 1$ without any change in shape.

4.3 Example 3: Finite Pulse

Let $a = 1$ and $u_0(x) = \begin{cases} 1 & 0 \le x \le 1 \\ 0 & \text{otherwise} \end{cases}$.

Then: $u(x,t) = \begin{cases} 1 & t \le x \le 1 + t \\ 0 & \text{otherwise} \end{cases}$.

The pulse shifts to the right with time, maintaining its width of 1.

5 Domain of Dependence and Domain of Influence

Two important concepts arise from the structure of characteristics.

5.1 Domain of Dependence

Question: Which point(s) of the initial data does the solution at (x,t) depend on?

Answer: Only the **single point** $x_0 = x - at$ on the $t = 0$ axis. This is because only one characteristic passes through (x, t), and it originated at x_0.

Definition: Domain of Dependence

The **domain of dependence** of the point (x, t) is the set of initial data points that influence the solution at (x, t). For the linear advection equation, it is the single point $\{x - at\}$ on the $t = 0$ line.

5.2 Domain of Influence

Question: If we perturb the initial condition at the single point x_0, which future points (x, t) will be affected?

Answer: All points on the characteristic line $x = x_0 + at$ for $t > 0$.

Definition: Domain of Influence

The **domain of influence** of the point x_0 on the initial line is the set of all (x, t) with $t > 0$ that are affected by the value $u_0(x_0)$. For the linear advection equation, it is the ray $\{(x_0 + at,\ t)\ :\ t > 0\}$.

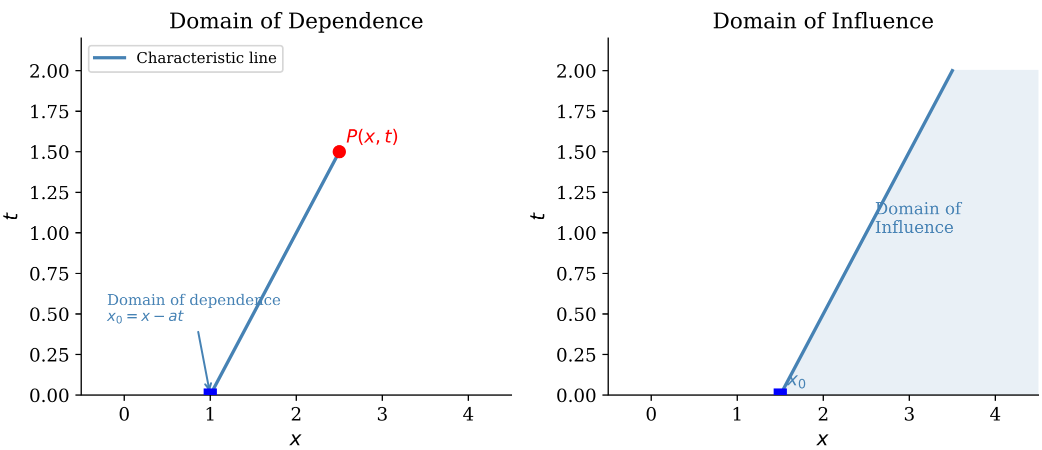

Figure 3: Left: The solution at $P = (x, t)$ depends only on the initial data at $x_0 = x - at$ (domain of dependence). Right: A perturbation at x_0 propagates along the characteristic and influences all future points on that ray (domain of influence).

5.3 A Simple Observation

The domain of dependence tells us something physically clear: the solution at any point (x, t) is decided entirely by what happened at one specific location in the past, namely $x_0 = x - at$. Nothing else matters.

This is a strong statement. It means that if we want to know u at some point in the future, we do not need to know the entire initial profile, we only need to know the initial data at one point. Information travels strictly along characteristics and nowhere else.

We will return to these ideas when we start solving these equations on a computer. For now, the key takeaway is: **characteristics are the paths along which information travels in a hyperbolic equation**.

6 Exercises

Exercises for Lecture 4

E1. Solve the initial value problem:

$$u_t + 3u_x = 0, \quad u(x,0) = e^{-x^2}$$

Write the solution $u(x,t)$ explicitly and sketch the profile at $t = 0$ and $t = 1$.

E2. For the advection equation $u_t + 2u_x = 0$ with $u(x,0) = \sin(2\pi x)$:

(a) Find the exact solution.

(b) What is the domain of dependence of the point $(3,1)$?

(c) What is the domain of influence of the point $x_0 = 0.5$?

E3. Consider the advection equation $u_t + au_x = 0$ with $a = -1.5$ (negative wave speed). Sketch the characteristic lines in the (x,t) plane and explain the direction of information propagation.

E4. Show that the general solution of $u_t + au_x = 0$ can be written as $u(x,t) = f(x - at)$ for any differentiable function f. Verify this by substituting back into the PDE.

E5. **(Boundary value problem)** Solve $u_t + u_x = 0$ on $x \in [0,\infty)$, $t > 0$ with:

$$u(x,0) = 0, \quad u(0,t) = \sin(t)$$

Hint: Think carefully about which boundary condition applies and for which region of the (x,t) domain.

A Computational Implementation (Julia)

A.1 Problem Statement

Solve the initial value problem for the 1D linear advection equation:

$$\frac{\partial u}{\partial t} + a\frac{\partial u}{\partial x} = 0, \qquad x \in [-2, 6], \quad t > 0 \tag{10}$$

with constant wave speed $a = 1.5$ and the smooth Gaussian initial profile:

$$u(x,0) = u_0(x) = \exp\left(-10(x - 0.5)^2\right) \tag{11}$$

A.2 Characteristic Method Formulation

For any evaluation point (x,t), the characteristic ray $\frac{dx}{dt} = a$ traces back to the initial coordinate:

$$x_0 = x - at \tag{12}$$

Since u is invariant along characteristics ($\frac{du}{dt} = 0$), the exact solution is directly obtained by evaluating the initial profile at x_0:

$$u(x,t) = u_0(x_0) = u_0(x - at) \tag{13}$$

A.3 Julia Code

The script below evaluates the exact solution on a spatial grid using the Method of Characteristics. The source code is maintained in `code/linear_advection_characteristics.jl`:

Listing 1: Julia Solver: Method of Characteristics for Linear Advection

```julia
# Linear Advection Equation: u_t + a*u_x = 0
using Plots

a = 1.5
u0(x) = exp(-10.0 * (x - 0.5)^2)
x = range(-1.0, 5.0, length=200)

for t in 0.0:0.04:2.0
    u = [u0(xi - a * t) for xi in x]
    display(plot(x, u, ylims=(0, 1.2), xlabel="x", ylabel="u(x,t)",
                 title="Linear Advection: t = $(round(t, digits=2))", legend=
                    false))
    sleep(0.02)
end
```

A.4 Running the Code

Executing the script displays a dynamic plot window that animates the Gaussian profile translating across the spatial domain $x \in [-1, 5]$ from $t = 0.0$ to $t = 2.0$:

Julia Terminal Output

```
$ julia code/linear_advection_characteristics.jl
Running live animation: Linear Advection (a = 1.5)...
Simulation finished.
```

References

[1] John D. Anderson. *Computational Fluid Dynamics: The Basics with Applications*. McGraw-Hill, New York, NY, 1995.

[2] Randall J. LeVeque. *Finite Volume Methods for Hyperbolic Problems*. Cambridge University Press, Cambridge, UK, 2002.