MA4110/MA5020: Computational Methods for Fluid Flow

Lecture 3: Classification of Partial Differential Equations

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1 Introduction: Why Classify PDEs?

In previous lectures, we saw that fluid motion is described by physical conservation laws expressed as partial differential equations (PDEs). Before constructing a computer solver, we must ask a fundamental question:

Why do we need to classify PDEs in Computational Fluid Dynamics?

The mathematical classification of a PDE dictates:

1. **How Information Travels:** Does physical information propagate at a finite wave speed, diffuse infinitely fast, or influence the entire domain globally?
2. **What Boundary Conditions are Allowed:** Posing incorrect boundary conditions for a given PDE type leads to ill-posed mathematical problems where solutions do not exist or blow up.
3. **Which Numerical Method to Use:** Hyperbolic problems require directional upwind schemes, parabolic problems require central differencing and implicit time-stepping, and elliptic problems require global matrix solvers.

Today's lecture provides a thorough classification of both second-order and first-order PDEs.

2 First-Order vs. Second-Order PDEs

The **order** of a PDE is defined as the order of the highest partial derivative present in the equation.

2.1 First-Order PDEs

A general first-order PDE in two independent variables (x, t) has the form:

$$F\left(x, t, u, \frac{\partial u}{\partial x}, \frac{\partial u}{\partial t}\right) = 0 \tag{1}$$

Fluid Flow Example: Advection Equation

Consider a pollutant tracer of concentration $u(x, t)$ carried by a fluid moving at speed a:

$$\frac{\partial u}{\partial t} + a\frac{\partial u}{\partial x} = 0 \tag{2}$$

Because the highest derivative is $\frac{\partial u}{\partial t}$ or $\frac{\partial u}{\partial x}$ (first derivative), this is a **first-order PDE**.

2.2 Second-Order PDEs

A general second-order PDE in two independent variables (x, y) has the form:

$$F\left(x, y, u, \frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial^2 u}{\partial x^2}, \frac{\partial^2 u}{\partial x \partial y}, \frac{\partial^2 u}{\partial y^2}\right) = 0 \tag{3}$$

Fluid Flow Example: Viscous Diffusion

Consider temperature or velocity diffusion due to fluid viscosity ν:

$$\frac{\partial u}{\partial t} = \nu \frac{\partial^2 u}{\partial x^2} \tag{4}$$

Because the highest derivative is $\frac{\partial^2 u}{\partial x^2}$ (second derivative), this is a **second-order PDE**.

3 Mathematical Classification of Second-Order PDEs

Consider the general quasi-linear second-order PDE in two independent variables (x, y):

$$A\frac{\partial^2 u}{\partial x^2} + B\frac{\partial^2 u}{\partial x \partial y} + C\frac{\partial^2 u}{\partial y^2} + D\frac{\partial u}{\partial x} + E\frac{\partial u}{\partial y} + Fu = G \tag{5}$$

where A, B, C, D, E, F, G may depend on x, y, u, u_x, u_y, but not on the second derivatives.

The classification of this equation is determined entirely by the **discriminant**:

$$\Delta = B^2 - 4AC \tag{6}$$

Classification of Second-Order PDEs

1. **Elliptic** ($B^2 - 4AC < 0$):
   - *Prototype Equation:* Laplace Equation $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \quad (A = 1, B = 0, C = 1 \implies \Delta = -4 < 0)$.
   - *Physical Behavior:* Represents steady-state equilibrium. Information propagates in all directions globally and instantaneously. A change at any boundary point affects the solution throughout the entire domain.
   - *Example:* Steady incompressible potential flow, Poisson equation for pressure $\nabla^2 p = S$.
2. **Parabolic** ($B^2 - 4AC = 0$):
   - *Prototype Equation:* Unsteady Heat/Diffusion Equation $\frac{\partial u}{\partial t} - \alpha \frac{\partial^2 u}{\partial x^2} = 0 \quad (A = -\alpha, B = 0, C = 0 \implies \Delta = 0)$.
   - *Physical Behavior:* Represents unsteady diffusion and dissipation processes. Signals travel continuously forward in time with infinite propagation speed (dissipating over time).
   - *Example:* Viscous boundary layers, unsteady heat conduction in fluids.
3. **Hyperbolic** ($B^2 - 4AC > 0$):
   - *Prototype Equation:* Wave Equation $\frac{\partial^2 u}{\partial t^2} - c^2 \frac{\partial^2 u}{\partial x^2} = 0 \quad (A = -c^2, B = 0, C = 1 \implies \Delta = 4c^2 > 0)$.
   - *Physical Behavior:* Represents wave propagation along real characteristic lines with a finite signal speed c.
   - *Example:* Unsteady compressible flow, acoustic waves.

4 Types of First-Order PDEs

First-order equations of the form $F(x, t, u, u_x, u_t) = 0$ are classified into three distinct categories based on how the unknown function u and its derivatives are combined.

4.1 Linear First-Order PDEs

A first-order PDE is **linear** if the unknown u and its derivatives $\frac{\partial u}{\partial t}, \frac{\partial u}{\partial x}$ appear to the first power only, and their coefficients depend only on the independent variables (x, t):

$$\frac{\partial u}{\partial t} + a(x, t)\frac{\partial u}{\partial x} = c(x, t)u + f(x, t) \tag{7}$$

Example (Linear Advection):

$$\frac{\partial u}{\partial t} + a\frac{\partial u}{\partial x} = 0 \quad (a = \text{constant}) \tag{8}$$

Here, the wave speed a is constant, so a wave profile translates across space without changing its shape.

4.2 Quasilinear First-Order PDEs

A first-order PDE is **quasilinear** if the highest derivatives ($\frac{\partial u}{\partial t}, \frac{\partial u}{\partial x}$) appear linearly (to the first power), but their coefficients depend on the unknown solution u itself:

$$\frac{\partial u}{\partial t} + a(x,t,u)\frac{\partial u}{\partial x} = c(x,t,u) \tag{9}$$

Example (Inviscid Burgers' Equation):

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = 0 \tag{10}$$

Here, the coefficient multiplying $\frac{\partial u}{\partial x}$ is the dependent variable u itself, meaning the coefficient depends directly on the solution u.

4.3 Fully Non-linear First-Order PDEs

A first-order PDE is **fully non-linear** if the partial derivatives $\frac{\partial u}{\partial t}$ or $\frac{\partial u}{\partial x}$ appear non-linearly (e.g. squared, multiplied together, or inside transcendental functions):

$$\frac{\partial u}{\partial t} + \left(\frac{\partial u}{\partial x}\right)^2 = 0 \quad \text{or} \quad \frac{\partial u}{\partial t} + \sin\left(\frac{\partial u}{\partial x}\right) = 0 \tag{11}$$

5 Why is a Scalar First-Order PDE Hyperbolic?

Now let us address a critical question: **What is the mathematical classification of a scalar first-order PDE, and why?**

Consider the general scalar first-order advection equation:

$$\frac{\partial u}{\partial t} + a(x,t,u)\frac{\partial u}{\partial x} = 0 \tag{12}$$

To classify this equation, we examine how physical information travels through the space-time (x,t) domain.

5.1 The Total Time Derivative

Consider a moving curve $x(t)$ in the space-time plane. The total rate of change of $u(x(t),t)$ along this path is given by the chain rule:

$$\frac{du}{dt} = \frac{\partial u}{\partial t} + \left(\frac{dx}{dt}\right)\frac{\partial u}{\partial x} \tag{13}$$

5.2 Matching with the PDE

Comparing the chain rule expansion with our PDE $\frac{\partial u}{\partial t} + a\frac{\partial u}{\partial x} = 0$, we choose the speed of our moving path to match the coefficient a:

$$\frac{dx}{dt} = a(x,t,u) \tag{14}$$

Substituting $\frac{dx}{dt} = a$ into the total derivative yields:

$$\frac{du}{dt} = 0 \implies u(x,t) = \text{constant along } \frac{dx}{dt} = a \tag{15}$$

5.3 Why This Means Hyperbolic

1. **Real Characteristic Lines Exist:** The trajectories defined by $\frac{dx}{dt} = a(x, t, u)$ are **real curves** in space-time. These are called **characteristic lines**.

2. **Finite Signal Speed:** Physical information (the value of u) is strictly carried along these real characteristic lines at a **finite speed** a.

3. **Mathematical Definition of Hyperbolicity:** Any differential equation whose solutions propagate along **real characteristic curves at a finite wave speed** is classified as **HYPERBOLIC**.

Comparison of PDE Behaviors

- **Elliptic PDEs** (like $u_{xx} + u_{yy} = 0$): Have **no real characteristics**. Information spreads everywhere globally and instantaneously.

- **Parabolic PDEs** (like $u_t = \alpha u_{xx}$): Have vertical characteristics ($t = \text{const}$). Information spreads with **infinite speed**.

- **Second-Order Hyperbolic PDEs** (like $u_{tt} - c^2 u_{xx} = 0$): Have **two families of real characteristics** $\frac{dx}{dt} = \pm c$. Signals travel as waves at finite speed c in both directions.

- **Scalar First-Order PDEs** ($u_t + au_x = 0$): Have **one family of real characteristic lines** $\frac{dx}{dt} = a$. Information travels at **finite speed** $a \implies$ **HYPERBOLIC**.

6 Summary

In this lecture, we established:

- Second-order PDEs are classified as **Elliptic** ($\Delta < 0$), **Parabolic** ($\Delta = 0$), or **Hyperbolic** ($\Delta > 0$) based on their discriminant $B^2 - 4AC$.

- First-order PDEs are categorized as **Linear**, **Quasilinear**, or **Fully Non-linear**.

- A scalar first-order advection equation $u_t + au_x = 0$ is **Hyperbolic** because information travels along real characteristic lines $\frac{dx}{dt} = a$ at finite wave speed.

References

[1] John D. Anderson. *Computational Fluid Dynamics: The Basics with Applications*. McGraw-Hill, New York, NY, 1995.

[2] Charles Hirsch. *Numerical Computation of Internal and External Flows: The Fundamentals of Computational Fluid Dynamics*. Butterworth-Heinemann, Oxford, UK, 2nd edition, 2007.