MA5020: Computational Methods for Fluid Flow

Lecture 2: The Dynamics of Flow: Euler and Navier-Stokes

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1 Introduction

In Lecture 1, we established our foundational framework and derived the *Continuity Equation*. Today, we follow the same step-by-step procedure to derive the conservation of **Momentum** and **Energy**.

By the end of this lecture, we will group these building blocks into the two most important systems in CFD: the **Euler Equations** and the **Navier-Stokes Equations**.

2 Conservation of Momentum

Physical Principle: Newton's Second Law ($\boldsymbol{F} = \frac{d\boldsymbol{P}}{dt}$) states that the rate of change of momentum of a system equals the sum of forces acting on it.

Step 1: Apply RTT Setting $\beta = \boldsymbol{v}$ (momentum per unit mass) in the Reynolds Transport Theorem:

$$\left.\frac{d\boldsymbol{P}}{dt}\right|_{sys} = \frac{\partial}{\partial t}\iiint_{CV} \rho\boldsymbol{v}\, dV + \iint_{CS} (\rho\boldsymbol{v})(\boldsymbol{v}\cdot\boldsymbol{n})\, dA \tag{1}$$

Step 2: Identify Forces For our model, forces consist of pressure p, viscous stress $\boldsymbol{\tau}$, and gravity $\boldsymbol{g}$:

$$\sum \boldsymbol{F} = \iint_{CS} (-p\boldsymbol{I} + \boldsymbol{\tau})\cdot\boldsymbol{n}\, dA + \iiint_{CV} \rho\boldsymbol{g}\, dV \tag{2}$$

Step 3: The Differential Form Using the Divergence Theorem and equating the terms:

$$\frac{\partial(\rho\boldsymbol{v})}{\partial t} + \nabla\cdot(\rho\boldsymbol{v}\otimes\boldsymbol{v} + p\boldsymbol{I} - \boldsymbol{\tau}) = \rho\boldsymbol{g} \tag{3}$$

3 Conservation of Energy

Physical Principle: The First Law of Thermodynamics ($dE = \delta Q - \delta W$) states that total energy changes due to heat added and work done.

Step 1: Define Total Energy Let $\beta = E_t = e + \frac{1}{2}v^2$ (internal + kinetic energy).

Step 2: Define Work and Heat Work is done by pressure and viscous forces ($\dot{W}$), and heat enters via conduction ($\boldsymbol{q}$):

$$\text{Work rate:} \quad \iint_{CS} (p\boldsymbol{I} - \boldsymbol{\tau})\cdot\boldsymbol{v}\cdot\boldsymbol{n}\, dA, \quad \text{Heat rate:} \quad \iint_{CS} \boldsymbol{q}\cdot\boldsymbol{n}\, dA \tag{4}$$

Step 3: Substitute into RTT and Result

$$\frac{\partial(\rho E_t)}{\partial t} + \nabla\cdot[\boldsymbol{v}(\rho E_t + p) - \boldsymbol{\tau}\cdot\boldsymbol{v} + \boldsymbol{q}] = 0 \tag{5}$$

4 Defining the Euler Equations (Inviscid System)

The **Euler Equations** are the core focus of this course. They represent the "Inviscid Limit" where we ignore friction ($\boldsymbol{\tau} = 0$) and heat conduction ($\boldsymbol{q} = 0$).

The Unified Euler System

$$\text{Mass:} \quad \frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \boldsymbol{v}) = 0 \tag{6}$$

$$\text{Momentum:} \quad \frac{\partial (\rho \boldsymbol{v})}{\partial t} + \nabla \cdot (\rho \boldsymbol{v} \otimes \boldsymbol{v} + p\boldsymbol{I}) = \rho \boldsymbol{g} \tag{7}$$

$$\text{Energy:} \quad \frac{\partial (\rho E_t)}{\partial t} + \nabla \cdot [\boldsymbol{v}(\rho E_t + p)] = 0 \tag{8}$$

5 Defining the Navier-Stokes Equations (Viscous System)

When we add the viscous stress tensor $\boldsymbol{\tau}$ and heat conduction $\boldsymbol{q}$ back into the Euler equations, we get the complete physical model of a real fluid.

The Full Navier-Stokes Model

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \boldsymbol{v}) = 0 \tag{9}$$

$$\frac{\partial (\rho \boldsymbol{v})}{\partial t} + \nabla \cdot (\rho \boldsymbol{v} \otimes \boldsymbol{v} + p\boldsymbol{I} - \boldsymbol{\tau}) = \rho \boldsymbol{g} \tag{10}$$

$$\frac{\partial (\rho E_t)}{\partial t} + \nabla \cdot [\boldsymbol{v}(\rho E_t + p) - \boldsymbol{\tau} \cdot \boldsymbol{v} + \boldsymbol{q}] = 0 \tag{11}$$

6 The Unified Vector Form (1D)

To prepare for computation, we often group the conserved variables into a vector $\mathbf{q}$. In 1D:

$$\frac{\partial \mathbf{q}}{\partial t} + \frac{\partial \mathbf{f}(\mathbf{q})}{\partial x} = \frac{\partial \mathbf{f}_v(\mathbf{q})}{\partial x} \tag{12}$$

where $\mathbf{f}$ is the inviscid flux (Euler) and $\mathbf{f}_v$ contains the viscous additions.

7 Summary

We have now derived every governing equation required for this course.

1. **Continuity:** Mass is constant.
2. **Euler:** Simplified physics (inviscid, adiabatic); captures wave physics.
3. **Navier-Stokes:** Complete physical model (viscous, conducting).

→ **Lecture 3:** We move from Physics to **Mathematics**. We will classify these PDEs to understand why we solve them the way we do.