MA5020: Computational Methods for Fluid Flow

Lecture 1: Foundations and the Continuity Equation

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1 Introduction: The Goal of Fluid Modeling

In this course, our ultimate goal is to solve the equations that govern fluid motion. These equations, known as the **Euler Equations** and the **Navier-Stokes Equations**, are the mathematical descriptions of universal conservation laws.

Before we can solve them on a computer, we must understand exactly where they come from. We begin today with the philosophical foundations and the derivation of the first building block: *Mass Conservation.*

2 The Continuum Hypothesis

We treat the fluid as a *continuum*. Instead of tracking individual molecules, we assume properties like density (ρ), pressure (p), and velocity ($\boldsymbol{v}$) vary smoothly in space. This allows us to use *Calculus* to describe the fluid.

3 Kinematic Descriptions: Lagrangian vs. Eulerian

How do we track a fluid parcel?

3.1 The Lagrangian Viewpoint (The Moving Observer)

We follow a specific group of particles. A property ϕ is a function of the particle's identity and time: $\phi = \phi(\boldsymbol{X}_0, t)$.

3.2 The Eulerian Viewpoint (The Fixed Observer)

We fix our gaze at a point in space $\boldsymbol{x}$ and watch the fluid flow past. This is the *standard for CFD* because it allows us to define a fixed grid.

4 The *Material Derivative*: Bridging the Gap

The *Material Derivative* ($\frac{D}{Dt}$) relates the Eulerian (fixed) viewpoint to the change felt by a moving parcel.

Step 1: The Total Differential Consider a property $\phi(x, y, z, t)$. Its total change $d\phi$ is:

$$d\phi = \frac{\partial \phi}{\partial t}dt + \frac{\partial \phi}{\partial x}dx + \frac{\partial \phi}{\partial y}dy + \frac{\partial \phi}{\partial z}dz \tag{1}$$

Step 2: Division by dt

$$\frac{d\phi}{dt} = \frac{\partial \phi}{\partial t} + \frac{\partial \phi}{\partial x}\left(\frac{dx}{dt}\right) + \frac{\partial \phi}{\partial y}\left(\frac{dy}{dt}\right) + \frac{\partial \phi}{\partial z}\left(\frac{dz}{dt}\right) \tag{2}$$

Step 3: Definition of Velocity Recognizing that $u = dx/dt$, $v = dy/dt$, and $w = dz/dt$:

$$\boxed{\frac{D\phi}{Dt} = \underbrace{\frac{\partial \phi}{\partial t}}_{\text{Local Change}} + \underbrace{(\boldsymbol{v} \cdot \nabla)\phi}_{\text{Convective Change}}} \tag{3}$$

5 Reynolds Transport Theorem (RTT)

The RTT is the tool we use to apply physics (which applies to a *System* of particles) to a fixed region of space (the *Control Volume, CV*).

Let B be a property (Mass, Momentum, Energy) and $\beta = B/m$.

$$\boxed{\left.\frac{dB}{dt}\right|_{sys} = \underbrace{\frac{\partial}{\partial t}\iiint_{CV} \rho\beta\, dV}_{\text{Local Change}} + \underbrace{\iint_{CS} \rho\beta(\boldsymbol{v}\cdot\boldsymbol{n})\, dA}_{\text{Net Flux Across Boundaries}} \qquad (4)}$$

6 Deriving the Continuity Equation (Mass Conservation)

Principle: The mass of a fluid system is constant: $\frac{dm}{dt} = 0$.

Step 1: Define Variables For mass, $B = m$, so $\beta = m/m = 1$.

Step 2: Substitute into RTT

$$\frac{\partial}{\partial t}\iiint_{CV} \rho\, dV + \iint_{CS} \rho(\boldsymbol{v}\cdot\boldsymbol{n})\, dA = 0 \tag{5}$$

This is the *Integral Form* of the Continuity Equation.

Step 3: Divergence Theorem To find the local law (PDE), we convert the surface flow into a volume integral:

$$\iint_{CS} \rho\boldsymbol{v}\cdot\boldsymbol{n}\, dA = \iiint_{CV} \nabla\cdot(\rho\boldsymbol{v})\, dV \tag{6}$$

Step 4: The Final PDE Combining the integrals and assuming the volume is arbitrary:

$$\boxed{\frac{\partial\rho}{\partial t} + \nabla\cdot(\rho\boldsymbol{v}) = 0 \qquad (7)}$$

7 Summary and Preview

We have laid the groundwork. We established the *Eulerian* framework, defined the *Material Derivative*, and derived our first governing equation: *Conservation of Mass.*

→ **Next Lecture:** We will follow these exact same steps to derive **Momentum** and **Energy** conservation, completing the full system of Euler and Navier-Stokes equations.