Foundational Mathematics for ML (MA2221)

Assignment (SVD & Low Rank Approximation)

Instructor: **Biswarup Biswas**

Instructions:

- All problems must be solved using **NumPy**.
- Use **functions, loops, and conditional statements**.
- Avoid using advanced routines beyond `np.linalg.svd`.
- Problems are marked with difficulty levels: Easy ([E]), Moderate ([M]), Challenging ([C]).

1. [E] Given
$$A = \begin{bmatrix} 3 & 2 & 2 \\ 2 & 3 & -2 \end{bmatrix},$$
   (a) Compute the Singular Value Decomposition of A.
   (b) Verify numerically that $A = U\Sigma V^T$.
   (c) Print U, the singular values, and V^T.

2. [M] Reconstruct the matrix using the rank–1 expansion
$$A = \sum_{i=1}^{r} \sigma_i u_i v_i^T.$$
   (a) Use a `for` loop to add one rank–1 term at a time.
   (b) Print the intermediate matrix after each iteration.
   (c) Compute the Frobenius norm of the reconstruction error.

3. [M] Using only the first $k = 1$ singular value:
   (a) Construct the rank–1 approximation A_1.
   (b) Compute $\|A - A_1\|_F$.
   (c) Compare it with the full reconstruction error.

4. [M] Energy retained by first k singular values is
$$\frac{\sum_{i=1}^{k} \sigma_i^2}{\sum_{i=1}^{r} \sigma_i^2}.$$
   (a) Use a `for` loop to compute cumulative energy.
   (b) Use an `if` statement to stop when energy exceeds 90%.
   (c) Print the smallest k satisfying this condition.

5. [E] Write a program that:
   (a) Sets singular values smaller than 0.5 to zero using a loop and `if`.

(b) Reconstructs the matrix.

(c) Computes the new reconstruction error.

6. [M] Load a grayscale image and compute its SVD.
   (a) Reconstruct the image using $k = 5, 20, 50$ (use a loop over k).
   (b) Use an `if` condition to label:
   - "Highly Compressed" if $k < 10$,
   - "Moderate Compression" if $10 \leq k < 40$,
   - "High Quality" if $k \geq 40$.

7. [M] Write a function

   `auto_rank(S, tol)`

   that:
   - counts singular values greater than `tol`,
   - uses a `for` loop and `if` statement,
   - returns the numerical rank.

8. [C] Verify numerically that

$$\|A\|_F^2 = \sum_{i=1}^{r} \sigma_i^2.$$

   (a) Compute both sides.

   (b) Compare the values up to machine precision.

   (c) Explain why this identity holds.

::: End of Assignment Sheet :::